

## Matrices Aumentadas

### Matrices Equivalentes por filas

#### Producción de matrices equivalentes por filas

Una matriz Aumentada se transforma en una matriz equivalente por filas si:

1. Se intercambian dos filas ( $f_i \leftrightarrow f_j$ )
2. Se multiplica una fila por una constante no nula ( $k f_i \rightarrow f_i$ )
3. Se agrega un múltiplo constante de una fila a otra fila dada. ( $f_i + k f_j \rightarrow f_i$ )

Observación: La flecha → significa "reemplaza a"

Solución de sistemas lineales mediante matrices aumentadas.

Ejemplo:

Resolver usando matrices aumentadas.

$$\begin{aligned} 3x_1 + 4x_2 &= 1 \\ x_1 - 2x_2 &= 7 \end{aligned} \quad (1)$$

Solución:

Se comienza por escribir la matriz aumentada correspondiente a (1)

$$\left[ \begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array} \right] \quad (2)$$

El objetivo consiste en usar las operaciones por filas mencionadas para intentar transformar la matriz (2) en la forma:

$$\left[ \begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array} \right] \quad (3)$$

en la que  $m$  y  $n$  son números reales. La solución del sistema (1) será evidente entonces, ya que (3) será la matriz aumentada del siguiente sistema:

$$\begin{aligned} x_1 &= m \\ x_2 &= n \end{aligned} \quad (4)$$

Se procede ahora a usar las operaciones por filas para transformar la matriz (2) en la forma (3)

1. Para obtener el 1 de la esquina superior izquierda, se intercambian las filas 1 y 2

$$\left[ \begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array} \right] \quad f_1 \leftrightarrow f_2 \quad \left[ \begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array} \right]$$

2. Para obtener un 0 en la esquina inferior izquierda, se multiplica  $f_1$  por  $(-3)$  y se suma a  $f_2$ , esto modifica a  $f_2$ , pero no a  $f_1$ , algunos encuentran útil escribir  $(-3)f_1$  fuera de la matriz para prevenir errores aritméticos, como se ilustra a continuación:

$$\left[ \begin{array}{ccc|c} 3 & 6 & -21 & \\ 1 & -2 & 7 & \\ 3 & 4 & 1 & \end{array} \right] \quad \overbrace{(-3)f_1 + f_2 \rightarrow f_2} \quad \left[ \begin{array}{ccc|c} & & & \\ 1 & -2 & 7 & \\ 0 & 10 & -20 & \end{array} \right]$$

3. Para obtener un 1 en la segunda fila, segunda columna ( $a_{22}$ ), se multiplica  $f_2$  por  $1/10$

$$\left[ \begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array} \right] \quad \frac{1}{10}f_2 \rightarrow f_2 \quad \left[ \begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array} \right]$$

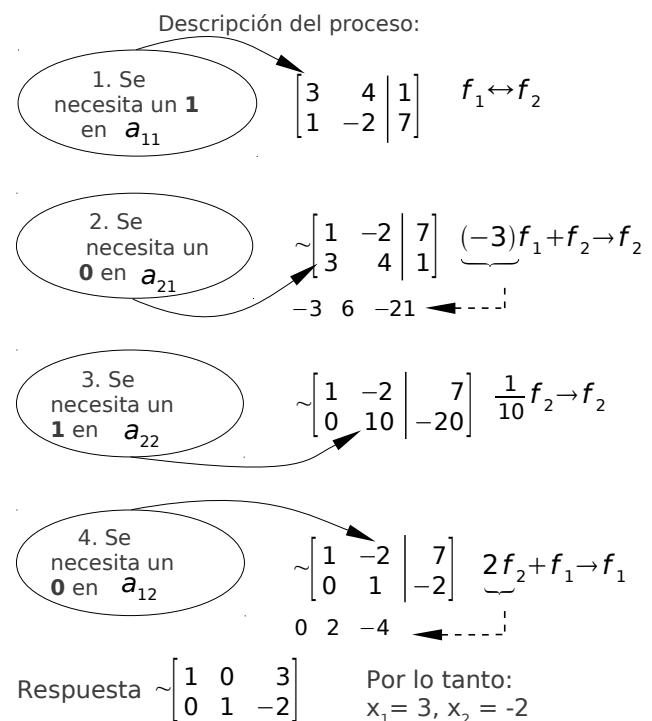
4. Para obtener un 0 en la primera fila, segunda columna ( $a_{12}$ ), se multiplica 2 por  $f_2$  y se suma el resultado a  $f_1$ , esto modifica a  $f_1$  pero no a  $f_2$

$$\left[ \begin{array}{ccc|c} 0 & 2 & -4 & \\ 1 & -2 & 7 & \\ 0 & 1 & -2 & \end{array} \right] \quad \overbrace{2f_2 + f_1 \rightarrow f_1} \quad \left[ \begin{array}{ccc|c} 1 & 0 & 3 & \\ 0 & 1 & -2 & \end{array} \right]$$

Con eso se logra el objetivo. La última es la matriz aumentada del sistema:

$$\begin{aligned} x_1 &= 3 \\ x_2 &= -2 \end{aligned} \quad (5)$$

Como el sistema (5) es equivalente al sistema (1), el problema queda resuelto, es decir  $x_1 = 3$  y  $x_2 = -2$ .



Resolver mediante la eliminación de Gauss – Jordan:

$$1) \begin{array}{l} 2x_1 + 4x_2 - 10x_3 = -2 \\ 3x_1 + 9x_2 - 21x_3 = 0 \\ x_1 + 5x_2 - 12x_3 = 1 \end{array}$$

$$2) \begin{array}{l} 3x_1 + 5x_2 - x_3 = -7 \\ x_1 + x_2 + x_3 = -1 \\ 2x_1 + 11x_3 = 7 \end{array}$$

$$3) \begin{array}{l} 3x_1 + 8x_2 - x_3 = -18 \\ 2x_1 + x_2 + 5x_3 = 8 \\ 2x_1 + 4x_2 + 2x_3 = -4 \end{array}$$

$$4) \begin{array}{l} 2x_1 + 7x_2 + 15x_3 = -12 \\ 4x_1 + 7x_2 + 13x_3 = -10 \\ 3x_1 + 6x_2 + 12x_3 = -9 \end{array}$$

$$5) \begin{array}{l} 2x_1 - x_2 - 3x_3 = 8 \\ x_1 - 2x_2 = 7 \end{array}$$

$$6) \begin{array}{l} 2x_1 + 4x_2 - 6x_3 = 10 \\ 3x_1 + 3x_2 - 3x_3 = 6 \end{array}$$

$$7) \begin{array}{l} 2x_1 + 3x_2 - x_3 = 1 \\ x_1 - 2x_2 + 2x_3 = -2 \end{array}$$

$$8) \begin{array}{l} x_1 - 3x_2 + 2x_3 = -1 \\ 3x_1 + 2x_2 - x_3 = 2 \end{array}$$

$$9) \begin{array}{l} 2x_1 + 2x_2 = 2 \\ x_1 + 2x_2 = 3 \\ -3x_2 = -6 \end{array}$$

$$10) \begin{array}{l} 2x_1 - x_2 = 0 \\ 3x_1 + 2x_2 = 7 \\ x_1 - x_2 = -1 \end{array}$$

$$11) \begin{array}{l} 2x_1 - x_2 = 0 \\ 3x_1 + 2x_2 = 7 \\ x_1 - x_2 = -2 \end{array}$$

$$12) \begin{array}{l} x_1 - 3x_2 = 5 \\ 2x_1 + x_2 = 3 \\ x_1 - 2x_2 = 5 \end{array}$$

$$13) \begin{array}{l} 3x_1 - 4x_2 - x_3 = 1 \\ 2x_1 - 3x_2 + x_3 = 1 \\ x_1 - 2x_2 + 3x_3 = 2 \end{array}$$

$$14) \begin{array}{l} 3x_1 + 7x_2 - x_3 = 11 \\ x_1 + 2x_2 - x_3 = 3 \\ 2x_1 + 4x_2 - 2x_3 = 10 \end{array}$$

$$15) \begin{array}{l} 2x_1 - 3x_2 + 3x_3 = -15 \\ 3x_1 + 2x_2 - 5x_3 = 19 \\ 5x_1 - 4x_2 - 2x_3 = -2 \end{array}$$

$$16) \begin{array}{l} 3x_1 - 2x_2 - 4x_3 = -8 \\ 4x_1 + 3x_2 - 5x_3 = -5 \\ 6x_1 - 5x_2 + 2x_3 = -17 \end{array}$$

$$17) \begin{array}{l} 5x_1 - 3x_2 + 2x_3 = 13 \\ 2x_1 + 4x_2 - 3x_3 = -19 \\ 4x_1 - 2x_2 + 5x_3 = 13 \end{array}$$

$$18) \begin{array}{l} 4x_1 - 2x_2 + 3x_3 = 0 \\ 3x_1 - 5x_2 - 2x_3 = -12 \\ 2x_1 + 4x_2 - 3x_3 = -4 \end{array}$$

$$19) \begin{array}{l} x_1 + 2x_2 - 4x_3 - x_4 = 7 \\ 2x_1 + 5x_2 - 9x_3 - 4x_4 = 16 \\ x_1 + 5x_2 - 7x_3 - 7x_4 = 13 \end{array}$$

$$20) \begin{array}{l} 2x_1 + 4x_2 + 5x_3 + 4x_4 = 8 \\ x_1 + 2x_2 + 2x_3 + x_4 = 3 \end{array}$$

$$21) \begin{array}{l} 2x_1 - 2x_2 + x_3 - 2x_4 = -4 \\ 2x_1 + 2x_2 - 3x_3 - 3x_4 = -1 \\ -x_1 + x_2 + 2x_3 + x_4 = 5 \\ x_2 - 2x_3 + x_4 = 0 \end{array}$$

$$22) \begin{array}{l} 2x_1 - 3x_2 + x_3 - 8x_4 = -2 \\ x_1 + 3x_2 + 2x_3 - x_4 = 5 \\ -x_1 + 2x_2 + x_3 + 3x_4 = 3 \\ 3x_2 + 2x_3 + 3x_3 - 7x_4 = 5 \end{array}$$

Tabla de Integrales

1. $\int u dv = uv - \int v du$	28. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2} + C$	50. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2} \ln \left  \frac{a+bu}{u} \right  + C$	73. $\int \sin^n u du = \frac{1}{n} \sin^{n-1} u \cos u + \frac{n-1}{n} \int \sin^{n-2} u du$
2. $\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$	29. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$	51. $\int \frac{u du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2} \ln  a+bu  + C$	74. $\int \cos^n u du = \frac{1}{n} \cos^{n-1} u \sin u + \frac{n-1}{n} \int \cos^{n-2} u du$
3. $\int \frac{du}{u} = \ln  u  + C$	30. $\int \sqrt{a^2-u^2} du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$	52. $\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2} \ln \left  \frac{a+bu}{u} \right  + C$	75. $\int \tan^n u du = \frac{1}{n-1} \tan^{n-1} u - \int \tan^{n-2} u du$
4. $\int e^u du = e^u + C$	31. $\int u^2 \sqrt{a^2-u^2} du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8} \sin^{-1} \frac{u}{a} + C$	53. $\int \frac{u^2 du}{(a+bu)^2} = \frac{1}{b^3} \left( a+bu - \frac{a^2}{a+bu} - 2a \ln  a+bu  \right) + C$	76. $\int \cot^n u du = \frac{-1}{n-1} \cot^{n-1} u - \int \cot^{n-2} u du$
5. $\int a^u du = \frac{a^u}{\ln a} + C$	32. $\int \frac{\sqrt{a^2-u^2}}{u} du = \sqrt{a^2-u^2} - a \ln \left  \frac{a+\sqrt{a^2-u^2}}{u} \right  + C$	54. $\int u \sqrt{a+bu} du = \frac{2}{15b^2} (3bu-2a)(a+bu)^{3/2} + C$	77. $\int \sec^n u du = \frac{1}{n-1} \tan u \sec^{n-2} u + \frac{n-2}{n-1} \int \sec^{n-2} u du$
6. $\int \sin u du = -\cos u + C$	33. $\int \frac{\sqrt{a^2-u^2}}{u^2} du = -\frac{1}{4} \sqrt{a^2-u^2} - \sin^{-1} \frac{u}{a} + C$	55. $\int \frac{udu}{\sqrt{a+bu}} = \frac{2}{3b^2} (bu-2a)\sqrt{a+bu} + C$	78. $\int \csc^n u du = \frac{-1}{n-1} \cot u \csc^{n-2} u + \frac{n-2}{n-1} \int \csc^{n-2} u du$
7. $\int \cos u du = \sin u + C$	34. $\int \frac{u^2 du}{\sqrt{a^2-u^2}} = -\frac{u}{2} \sqrt{a^2-u^2} - \frac{a^2}{a} \sin^{-1} \frac{u}{a} + C$	56. $\int \frac{u^2 du}{\sqrt{a+bu}} = \frac{2}{15b^3} (8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$	79. $\int \sin a u \sin b u du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$
8. $\int \sec^2 u du = \tan u + C$	35. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a} \ln \left  \frac{a+\sqrt{a^2-u^2}}{u} \right  + C$	57. $\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}} \ln \left  \frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}} \right  + C, \text{ si } a > 0$	80. $\int \cos a u \cos b u du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$
9. $\int \csc^2 u du = -\cot u + C$	36. $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u} \sqrt{a^2-u^2} + C$	$\downarrow$ $= \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bu}{-a}} + C, \text{ si } a < 0$	81. $\int \sin a u \cos b u du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$
10. $\int \sec u \cdot \tan u du = \sec u + C$	37. $\int (a^2-u^2)^{3/2} du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8} \sin^{-1} \frac{u}{a} + C$	58. $\int \frac{\sqrt{a+bu}}{u} du = 2\sqrt{a+bu} + a \int \frac{du}{u\sqrt{a+bu}}$	83. $\int u \cos u du = \cos u + u \sin u + C$
11. $\int \csc u \cdot \cot u du = -\csc u + C$	38. $\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$	59. $\int \frac{\sqrt{a+bu}}{u^2} du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2} \int \frac{du}{u\sqrt{a+bu}}$	84. $\int u^n \sin u du = -u^n \cos u + n \int u^{n-1} \cos u du$
12. $\int \tan u du = \ln  \sec u  + C$	39. $\int \sqrt{u^2-a^2} du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2} \ln  u+\sqrt{u^2-a^2}  + C$	60. $\int u^n \sqrt{a+bu} du = \frac{2}{b(2n+3)} \left[ u^n (a+bu)^{3/2} - na \int u^{n-1} \sqrt{a+bu} du \right]$	85. $\int u^n \cos u du = u^n \sin u - n \int u^{n-1} \sin u du$
13. $\int \cot u du = \ln  \sin u  + C$	40. $\int u^2 \sqrt{u^2-a^2} du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8} \ln  u+\sqrt{u^2-a^2}  + C$	61. $\int \frac{u^n du}{\sqrt{a+bu}} = \frac{2u^n \sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)} \int \frac{u^{n-1} du}{\sqrt{a+bu}}$	86. $\int \sin^n u \cos^m u du = (2 \text{ result. equivalentes})$
14. $\int \sec u du = \ln  \sec u + \tan u  + C$	41. $\int \frac{\sqrt{u^2-a^2}}{u} du = \sqrt{u^2-a^2} - a \cos^{-1} \frac{a}{ u } + C$	62. $\int \frac{du}{u^n \sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)} \int \frac{du}{u^{n-1} \sqrt{a+bu}}$	$\uparrow$ $= \frac{\sin^{n-1} \cos^{m+1} u}{n+m} + \frac{n-1}{n+m} \int \sin^{n-2} u \cos^m u du$
15. $\int \csc u du = \ln  \csc u - \cot u  + C$	42. $\int \frac{\sqrt{u^2-a^2}}{u^2} du = -\frac{\sqrt{u^2-a^2}}{u} + \ln  u+\sqrt{u^2-a^2}  + C$	63. $\int \sin^2 u du = \frac{1}{2}u - \frac{1}{4} \sin 2u + C$	87. $\int \sin^{-1} u du = u \sin^{-1} u + \sqrt{1-u^2} + C$
16. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1} \frac{u}{a} + C$	43. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln  u+\sqrt{u^2-a^2}  + C$	64. $\int \cos^2 u du = \frac{1}{2}u + \frac{1}{4} \sin 2u + C$	88. $\int \cos^{-1} u du = u \cos^{-1} u - \sqrt{1-u^2} + C$
17. $\int \frac{du}{a^2+u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$	44. $\int \frac{u^2 du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2} \ln  u+\sqrt{u^2-a^2}  + C$	65. $\int \tan^2 u du = \tan u - u + C$	89. $\int \tan^{-1} u du = u \tan^{-1} u - \frac{1}{2} \ln(1+u^2) + C$
18. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$	45. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$	66. $\int \cot^2 u du = -\cot u - u + C$	90. $\int u \sin^{-1} u du = \frac{2u^2-1}{4} \sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$
19. $\int \frac{du}{a^2-u^2} = \frac{1}{2a} \ln \left  \frac{u+a}{u-a} \right  + C$	46. $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$	67. $\int \sin^3 u du = -\frac{1}{3}(2+\sin^2 u) \cos u + C$	91. $\int u \cos^{-1} u du = \frac{2u^2-1}{4} \cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$
20. $\int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln \left  \frac{u-a}{u+a} \right  + C$	47. $\int \frac{u du}{a+bu} = \frac{1}{b^2} (a+bu - a \ln  a+bu ) + C$	68. $\int \cos^3 u du = \frac{1}{3}(2+\cos^2 u) \sin u + C$	92. $\int u \tan^{-1} u du = \frac{u^2+1}{2} \tan^{-1} u - \frac{u}{2} + C$
21. $\int \sqrt{a^2+u^2} du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2} \ln(u+\sqrt{a^2+u^2}) + C$	48. $\int \frac{u^2 du}{a+bu} = \frac{1}{2b^3} [(a+bu)^2 - 4a(a+bu) + 2a^2 \ln  a+bu ] + C$	69. $\int \tan^3 u du = \frac{1}{2} \tan^2 u + \ln  \cos u  + C$	93. $\int u^n \sin^{-1} u du = \frac{1}{n+1} \left[ u^{n+1} \sin^{-1} u - \int \frac{u^{n+1} du}{\sqrt{1-u^2}} \right], n \neq -1$
22. $\int u^2 \sqrt{a^2+u^2} du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8} \ln(u+\sqrt{a^2+u^2}) + C$	70. $\int \cot^3 u du = -\frac{1}{2} \cot^2 u - \ln  \sin u  + C$		94. $\int u^n \cos^{-1} u du = \frac{1}{n+1} \left[ u^{n+1} \cos^{-1} u + \int \frac{u^{n+1} du}{\sqrt{1-u^2}} \right], n \neq -1$
23. $\int \frac{\sqrt{a^2+u^2}}{u} du = \sqrt{a^2+u^2} - a \ln \left  \frac{a+\sqrt{a^2+u^2}}{u} \right  + C$	71. $\int \sec^3 u du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln  \sec u + \tan u  + C$		
24. $\int \frac{\sqrt{a^2+u^2}}{u^2} du = -\frac{\sqrt{a^2+u^2}}{u} + \ln(a+\sqrt{a^2+u^2})$	72. $\int \csc^3 u du = -\frac{1}{2} \csc u \cot u + \frac{1}{2} \ln  \csc u + \cot u  + C$		
25. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln(u+\sqrt{a^2+u^2})$	73. $\int \frac{du}{u(a+bu)} = \frac{1}{a} \ln \left  \frac{u}{a+bu} \right  + C$		
26. $\int \frac{u^2 du}{\sqrt{a^2+u^2}} = \frac{u}{2} \sqrt{a^2+u^2} + C$			
27. $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a} \ln \left  \frac{\sqrt{a^2+u^2}+a}{u} \right  + C$			

$$95. \int u^n \tan^{-1} u \, du = \frac{1}{n+1} \left[ u^{n+1} \tan^{-1} u - \int \frac{u^{n+1} du}{\sqrt{1-u^2}} \right], n \neq -1$$

$$96. \int u e^{au} \, du = \frac{1}{a^2} (au - 1) e^{au} + C$$

$$97. \int u^n e^{au} \, du = \frac{1}{a} u^n e^{au} - \frac{n}{a} \int u^{n-1} e^{au} \, du$$

$$98. \int e^{au} \sin bu \, du = \frac{e^{au}}{a^2+b^2} (a \sin bu - b \cos bu) + C$$

$$99. \int e^{au} \cos bu \, du = \frac{e^{au}}{a^2+b^2} (a \cos bu + b \sin bu) + C$$

$$100. \int \ln u \, du = u \ln u - u + C$$

$$101. \int u^n \ln u \, du = \frac{u^{n+1}}{(n+1)^2} [(n+1) \ln u - 1] + C$$

$$102. \int \frac{1}{u \ln u} \, du = \ln |\ln u| + C$$

$$103. \int \operatorname{senh} u \, du = \cosh u + C$$

$$104. \int \cosh u \, du = \operatorname{senh} u + C$$

$$105. \int \tanh u \, du = \ln \cosh u + C$$

$$106. \int \coth u \, du = \ln |\operatorname{senh} u| + C$$

$$107. \int \operatorname{sech} u \, du = \tan^{-1} |\operatorname{senh} u| + C$$

$$108. \int \operatorname{csch} u \, du = \ln \left| \tanh \frac{1}{2} u \right| + C$$

$$109. \int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$110. \int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$111. \int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$112. \int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

$$113. \int \sqrt{2au-u^2} \, du = \frac{u-a}{2} + \sqrt{2au-u^2} + \frac{a^2}{2} \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$114. \int u \sqrt{2au-u^2} \, du = \frac{2u^2-au-3a^2}{6} \sqrt{2au-u^2} + \frac{a^3}{2} \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$115. \int \frac{\sqrt{2au-u^2}}{u} \, du = \sqrt{2au-u^2} - a \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$116. \int \frac{\sqrt{2au-u^2}}{u^2} \, du = -\frac{\sqrt{2au-u^2}}{u} - \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$117. \int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$118. \int u \frac{du}{\sqrt{2au-u^2}} = -\sqrt{2au-u^2} + a \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$119. \int u^2 \frac{du}{\sqrt{2au-u^2}} = -\frac{(u+3a)}{2} \sqrt{2au-u^2} + \frac{3a^2}{2} \cos^{-1} \left( \frac{a-u}{a} \right) + C$$

$$120. \int u \frac{du}{u \sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$$

### Bibliografía:

Stewart, James. **Cálculo, Conceptos y Contextos**. 2006. Editorial Thomson. 978 pp. Impreso en México.

**Todos proporcionan las mismas medidas angulares, el valor de  $180^\circ$  es la línea horizontal del instrumento.**

