Universidad Nacional José María Arguedas

Identidad y Excelencia para el Trabajo Productivo y el Desarrollo

FUNCIONES REALES DE VARIAS VARIABLES

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FUNCIONES REALES DE VARIAS VARIABLES

Definición:

Una función real de n variables independientes denotado por $f : D \subset R^2 \to B \subset R$ es una regla de correspondencia de un conjunto "D" de vectores del espacio n dimensional a un conjunto "B" de números reales talque:

$$f : D \subset R^2 \to B \subset R$$
$$\vec{x} \to f(x) = z \quad ; \quad \vec{x} = (x_1; x_2; ....; x_n)$$
$$\Rightarrow f(x_1; x_2; ......; x_n) = z$$

A las variables $x_1; x_2; ....; x_n$ se les llama variables independientes y a z se le llama variable dependiente.

Dominio de una función de varias variables:

Se llama dominio de definición o dominio de existencia de la función f al conjunto:

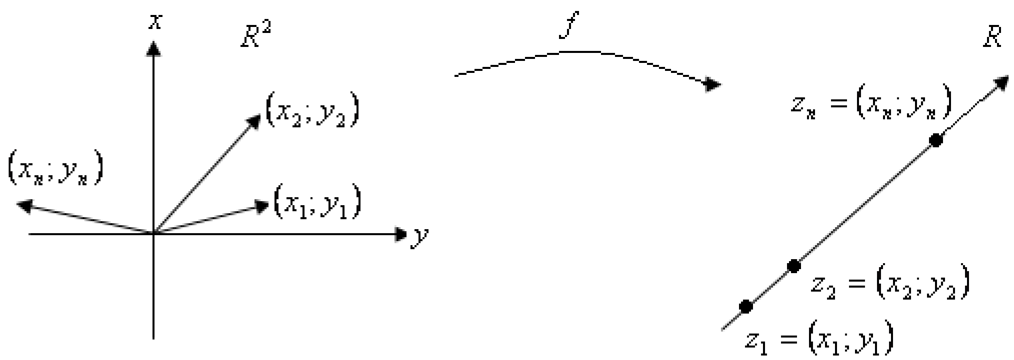
$$D_f = \{\vec{x} = (x_1; x_2; ......; x_n) \in R^n / z = f(\vec{x}) = f(x_1; x_2; ......; x_n)\}$$

Los casos más importantes para su estudio son las funciones reales de dos y tres variables, por lo tanto presentaremos los siguientes casos.

1º Caso: Si $f : R^2 \to R$ es una función real de dos variables independientes.

$$f : R^2 \to R$$
$$(x, y) \to f(x, y) = z$$

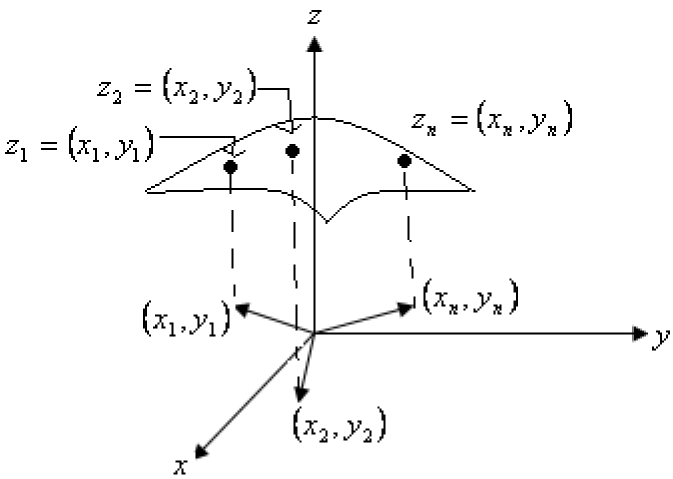
Gráficamente:



2º Caso: Si $E : R^3 \to R$ es una función de tres variables independientes.

S recibe la denominación de superficie, cuya ecuación es $E(x, y, z) = 0$, el cual define una o más funciones de la forma $z = f(x, y)$

Es decir: $E(x, y, z) = 0$ define implícitamente a la función $z = f(x, y) \quad (x_1, y_1)$



<u>EJEMPLOS</u>:

1. Describe y grafica el dominio de las siguientes funciones;
   a) $z = \sqrt{1 - x^2 - y^2}$
   b) $z = \sqrt{x^2 - 4} + \sqrt{4 - y^2}$

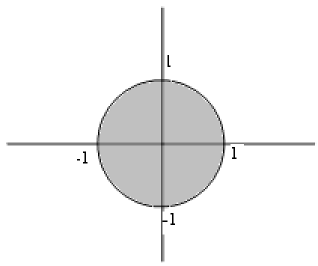
SOLUCIÓN:

a) $z = f(x,y) = \sqrt{1 - x^2 - y^2}$ existe $\Leftrightarrow 1 - x^2 - y^2 \geq 0$

$-x^2 - y^2 \geq -1 \Rightarrow x^2 + y^2 \leq 1$

$D_f = \{(x,y) \in R^2 / x^2 + y^2 \leq 1\}$

Gráficamente:



b) $z = f(x,y) = \sqrt{x^2 - 4} + \sqrt{4 - y^2}$

sea $g(x,y) = \sqrt{x^2 - 4}$; $h(x,y) = \sqrt{4 - y^2}$

$\Rightarrow f(x,y) = g(x,y) + h(x,y)$

$g(x,y) = \sqrt{x^2 - 4}$ existe $\Leftrightarrow x^2 - 4 \geq 0 \Rightarrow x \geq 2 \wedge x \leq -2$

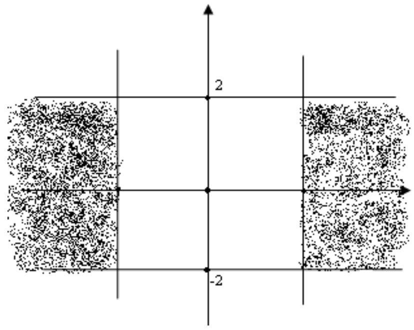
$\Rightarrow D_g = \{(x,y) \in R^2 / x \leq -2 \wedge x \geq 2\}$

$h(x,y) = \sqrt{4 - y^2}$ existe $\Leftrightarrow 4 - y^2 \geq 0 \Rightarrow -2 \leq y \leq 2$

$\Rightarrow D_h = \{(x,y) \in R^2 / -2 \leq y \leq 2\}$

$D_f = D_g \cap D_h = \{(x,y) \in R^2 / |x| \geq 2 \wedge |y| \leq 2\}$

Gráficamente:



2. **Hallar el dominio de** $z = \ln(36 - 4x^2 - 9y^2)$

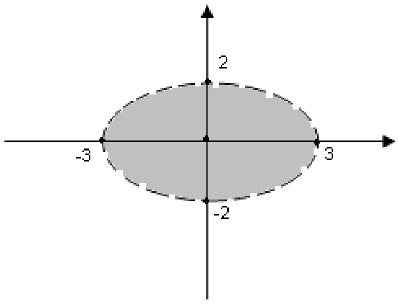
SOLUCIÓN:

$$z = f(x, y) = \ln(36 - 4x^2 - 9y^2) \text{ existe} \Leftrightarrow 36 - 4x^2 - 9y^2 > 0$$

$$\Rightarrow 4x^2 + 9y^2 < 36 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} < 1$$

$$D_f = \left\{ (x, y) \in R^2 / \frac{x^2}{9} + \frac{y^2}{4} < 1 \right\}$$

Gráficamente:



3. **Hallar el dominio de** $f(x, y) = \text{arcsen}\left(\frac{y-1}{x} \right)$

SOLUCIÓN:

$$z = f(x, y) = \text{arcsen}\left(\frac{y-1}{x} \right) \Rightarrow \text{senz} = \frac{y-1}{x}$$

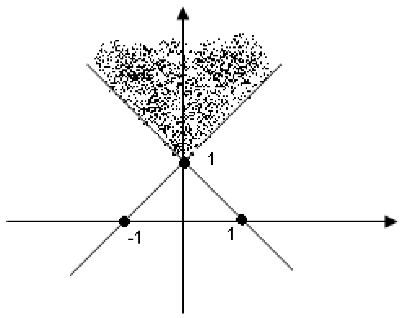
$$-1 \leq \text{senz} \leq 1$$

$$-1 \leq \frac{y-1}{x} \leq 1$$

a) Si $x > 0$, entonces $-x \leq y - 1 \leq x \Rightarrow -x + 1 \leq y \leq x + 1$

$$D_f = \{(x, y) \in R^2 / -x + 1 \leq y \leq x + 1\}$$

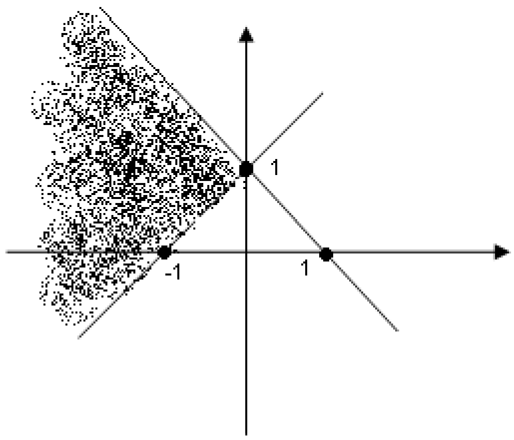
Gráficamente:



b) Si $x < 0$, entonces $-x \geq y - 1 \geq x \Rightarrow -x + 1 \geq y \geq x + 1$

$D_f = \{(x, y) \in R^2 \,/\, x + 1 \leq y \leq 1 - x\}$

Gráficamente:



4.

LÍMITE Y CONTINUIDAD DE FUNCIONES DE VARIAS VARIABLES

1. Límite:

Sea $f : S \subset R^2 \to R$ una función de dos variables independientes definida en el conjunto S y sea $\vec{a} = (a,b)$ un punto de acumulación de S entonces:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \Leftrightarrow \forall \varepsilon > 0; \exists \delta > 0 \,/\, |f(x,y) - L| < \varepsilon \text{ siempre que}$$
$$0 < |(x,y) - (a,b)| < \delta$$

O equivalentemente a:

$$\lim_{\substack{x\to a \\ y\to b}} f(x,y) = L \Leftrightarrow \forall \varepsilon > 0; \exists \delta > 0 \,/\, |f(x,y) - L| < \varepsilon \text{ siempre que}$$
$$0 < |x - a| < \delta \wedge 0 < |y - b| < \delta$$

Observación:

Al considerar $\lim_{x\to a} f(x)$ sabemos que el punto x se aproxima al punto a a lo largo del eje x por la derecha y por la izquierda respectivamente, en cambio en una función de dos variables independientes $z = f(x,y)$ el punto $\vec{x} = (x,y)$ se aproxima

al punto $\vec{a}=(a,b)$ a través de dos curvas que pasan por el punto $\vec{a}=(a,b)$ tales que:

$$\lim_{\substack{\vec{x}\to\vec{a}\\ \vec{x}\in C_1}} f\left(\vec{x}\right)=L_1 \qquad \wedge \qquad \lim_{\substack{\vec{x}\to\vec{a}\\ \vec{x}\in C_2}} f\left(\vec{x}\right)=L_2$$

Entonces:

i. Si $L_1 \neq L_2 \Rightarrow \lim_{\vec{x}\to\vec{a}} f\left(\vec{x}\right)$ no existe.

ii. Si $L_1 = L_2$ se considera una tercera curva C_3 que pasa por el punto $\vec{a}=(a,b)$ tal que:

$$\lim_{\substack{\vec{x}\to\vec{a}\\ \vec{x}\in C_3}} f\left(\vec{x}\right)=L_3=L_2=L_1$$

Se puede considerar que el $\lim_{\vec{x}\to\vec{a}} f\left(\vec{x}\right)$ existe y para verificarlo se debe probar dicho límite aplicando la definición.

2. Continuidad:
Se dice que la función $f: S\subset R^2 \to R$ es continua en el punto $P_0 \in S$ si se cumple las siguientes condiciones:
   i. $f(P_0)$ **existe**
   ii. $\lim_{\vec{x}\to P_0} f\left(\vec{x}\right)$ **existe**
   iii. $\lim_{\vec{x}\to P_0} f\left(\vec{x}\right)= f(P_0)$

Definición: **Se dice que una función $f: S\subset R^n \to R$ es continua en $P_0 \in S$ sí y sólo si cada una de las funciones coordenadas: $f_i : S\subset R \to R$; $\forall i=\overline{1,n}$ son continuas en el punto P_0.**

EJEMPLOS:

1. **Verificar los siguientes límites:**
   a) $\lim_{(x,y)\to(2,3)}(3x+2y)=12$
   b) $\lim_{(x,y)\to(1,1)}\left(x^2+y^2\right)=2$
   c) $\lim_{(x,y)\to(3,-1)}\left(x^2+y^2-4x+2y\right)=-4$
   d) $\lim_{(x,y)\to(3,-1)}\left(x^2+2xy\right)=3$

SOLUCIÓN

a) $\lim_{(x,y)\to(2,3)}(3x+2y)=12$

$\forall\varepsilon>0;\exists\delta\geq 0 / \left|f(x,y)-L\right|<\varepsilon$ **siempre que** $0<|x-a|<\delta \ \wedge \ 0<|y-b|<\delta$

$\forall\varepsilon>0;\exists\delta\geq 0 / \left|(3x+2y)-12\right|<\varepsilon$ **siempre que** $0<|x-2|<\delta \ \wedge \ 0<|y-3|<\delta$

De: $|3x+2y-12|=|(3x-6)+(2y-6)|=|3(x-2)+2(y-3)|$

Y como: $|A+B|=|A|+|B|$

$|3x+2y-12|=|3(x-2)+2(y-3)|\leq|3||x-2|+|2||y-3|\leq 3|x-2|+2|y-3|$

$$< 3\delta + 2\delta = 5\delta < \varepsilon \Rightarrow \delta = \frac{\varepsilon}{5}$$

Si $\delta = \frac{\varepsilon}{5}$**; entonces** $\lim\limits_{(x,y)\to(2,3)} (3x + 2y) = 12$

b) $\lim\limits_{(x,y)\to(1,1)} (x^2 + y^2) = 2$

$\forall \varepsilon > 0; \exists \delta \geq 0 / |f(x,y) - L| < \varepsilon$ **siempre que** $0 < |x - a| < \delta \wedge 0 < |y - b| < \delta$

$\forall \varepsilon > 0; \exists \delta \geq 0 / |(x^2 + y^2) - 2| < \varepsilon$ **siempre que**

$0 < |x - 1| < \delta \wedge 0 < |y - 1| < \delta$

De: $|x^2 + y^2 - 2| = |(x^2 - 1) + (y^2 - 1)|$

$$= |(x-1)(x+1) + (y+1)(y-1)|$$

$$\leq |x+1||x-1| + |y+1||y-1|$$

Debemos acotar superiormente los factores $|x+1|$ **y** $|y+1|$**, elegimos entonces** $\delta_1 = 1$

$|x - 1| < 1 \Rightarrow -1 < x - 1 < 1 \Rightarrow 0 < x < 2 \Rightarrow 1 < x + 1 < 3 \Rightarrow |x + 1| < 3$

$|y - 1| < 1 \Rightarrow -1 < y - 1 < 1 \Rightarrow 0 < y < 2 \Rightarrow 1 < y + 1 < 3 \Rightarrow |y + 1| < 3$

$\Rightarrow |(x+1)(x-1) + (y+1)(y-1)| \leq |x+1||x-1| + |y+1||y-1|$

$$\Rightarrow 3\delta + 3\delta = 6\delta < \varepsilon \Rightarrow \delta_2 = \frac{\varepsilon}{6}$$

Por lo tanto: $\delta = \min\{\delta_1; \delta_2\} = \min\left\{1; \frac{\varepsilon}{6}\right\}$

c) $\lim\limits_{(x,y)\to(3,-1)} (x^2 + y^2 - 4x + 2y) = -4$

$\forall \varepsilon > 0; \exists \delta > 0 / |f(x,y) - L| < \varepsilon$ **siempre que** $0 < |x - a| < \delta \wedge 0 < |y - b| < \delta$

$\forall \varepsilon > 0; \exists \delta > 0 / |(x^2 + y^2 - 4x + 2y) + 4| < \varepsilon$ **siempre que**

$0 < |x - 3| < \delta \wedge 0 < |y + 1| < \delta$

De: $|x^2 + y^2 - 4x + 2y + 4| = |(x^2 - 4x + 3) + (y^2 + 2y + 1)|$

$$= |(x-1)(x-3) + (y+1)(y+1)|$$

$$\leq |x-1||x-3| + |y+1||y+1|$$

Debemos acotar los factores $|x-1|$ **y** $|y+1|$**, elegimos entonces** $\delta_1 = 1$

$|x - 3| < 1 \Rightarrow -1 < x - 3 < 1 \Rightarrow 2 < x < 4 \Rightarrow 1 < x - 1 < 3$

$|y + 1| < 1$

$\Rightarrow |x + y^2 - 4x + 2y + 4| \leq |x-1||x-3| + |y+1||y+1|$

$$\Rightarrow 3\delta + \delta = 4\delta < \varepsilon \Rightarrow \delta_2 = \frac{\varepsilon}{4}$$

Por lo tanto: $\delta = \min\{\delta_1; \delta_2\} = \min\left\{1; \frac{\varepsilon}{4}\right\}$

d) $\lim\limits_{(x,y)\to(3,-1)} (x^2 + 2xy) = 3$

$\forall\varepsilon>0;\exists\delta>0 / |f(x,y)-L|<\varepsilon$ **siempre que** $0<|x-a|<\delta \wedge 0<|y-b|<\delta$

$\forall\varepsilon>0;\exists\delta>0 / |(x^2+2xy)-3|<\varepsilon$ **siempre que**

$0<|x-3|<\delta \wedge 0<|y+1|<\delta$

De: $|2x(y+1)+(x+1)(x-3)|=|x^2+2xy+2x-2x-3|$

$=|2x(y+1)+(x+1)(x-3)|\le|2x||y+1|+|x+1||x-3|$

Acotando $|2x|$ **y** $|y+1|$**, elegimos entonces** $\delta_1=1$

$\Rightarrow|x-3|<1\Rightarrow-1<x-3<1\Rightarrow 3<x+1<5$

$\Rightarrow|x-3|<1\Rightarrow-1<x-3<1\Rightarrow 2<x<4\Rightarrow 4<2x<8$

$\Rightarrow|2x||y+1|+|x+1||x-3|=8\delta+5\delta=13\delta<\varepsilon\Rightarrow\delta_2=\frac{\varepsilon}{13}$

Por lo tanto: $\delta=\min\{\delta_1;\delta_2\}=\min\left\{1;\frac{\varepsilon}{13}\right\}$

2. **Calcular los siguientes límites.**

a. $\lim\limits_{(x,y)\to(2,-5)}(x^2+xy-y)$

b. $\lim\limits_{(x,y)\to(1,2)}\dfrac{x^3y+\operatorname{sen}\left(\frac{\pi xy}{2}\right)}{\sqrt{xy^2-\cos(2\pi x)}}$

c. $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy\operatorname{sen}(\operatorname{sen}2xy)}{1-\cos(\operatorname{sen}4xy)}$

d. $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y^2}{x^4+y^4}$

e. $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y^2}{x^2+y^2}$

SOLUCIÓN:

a. $\lim\limits_{(x,y)\to(2,-5)}(x^2+xy-y)=9$

b. $\lim\limits_{(x,y)\to(1,2)}\dfrac{x^3y+\operatorname{sen}\left(\frac{\pi xy}{2}\right)}{\sqrt{xy^2-\cos(2\pi x)}}=\dfrac{2+\operatorname{sen}\pi}{\sqrt{4-\cos 2\pi}}=\dfrac{2}{\sqrt{4-1}}=\dfrac{2}{\sqrt{3}}$

c. $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy\operatorname{sen}(\operatorname{sen}2xy)}{1-\cos(\operatorname{sen}4xy)}=\lim\limits_{(x,y)\to(0,0)}\dfrac{xy\dfrac{\operatorname{sen}[\operatorname{sen}2xy]}{\operatorname{sen}xy}.\dfrac{\operatorname{sen}2xy}{2xy}.2xy}{\dfrac{1-\cos[\operatorname{sen}4xy]}{\operatorname{sen}^2 4xy}.\left(\dfrac{\operatorname{sen}4xy}{4xy}\right)^2.16x^2y^2}$

$\lim\limits_{(x,y)\to(0,0)}\dfrac{2x^2y^2}{8x^2y^2}=\dfrac{1}{4}$

d. $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y^2}{x^4+y^4}=\dfrac{0}{0}$ **indeterminado**

Entonces nos definimos: $C_1=\{(x,y)\in R^2 / y=x\}$ **y** $C_2=\{(x,y)\in R^2 / y=x^2\}$

Sea $f(x,y)=f(\vec{x})=\dfrac{x^2y^2}{x^4+y^4}$

1. $\lim\limits_{\substack{x\to 0\\ x\in C_1}} f(\bar{x}) = \lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y)\in C_1}} \dfrac{x^2 y^2}{x^4 + y^4} = \lim\limits_{x\to 0} \dfrac{x^4}{2x^4} = \dfrac{1}{2} = L_1$

2. $\lim\limits_{\substack{x\to 0\\ x\in C_2}} f(\bar{x}) = \lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y)\in C_2}} \dfrac{x^2 y^2}{x^4 + y^4} = \lim\limits_{x\to 0} \dfrac{x^6}{x^4 + x^8} = \lim\limits_{x\to 0} \dfrac{x^6}{x^4(1 + x^4)}$

$$= \lim_{x\to 0} \frac{x^2}{1 + x^4} = 0 = L_1$$

Como $L_1 \neq L_2 \Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y^2}{x^4 + y^4}$ no existe.

3. **Analizar la continuidad de los siguientes limites:**

a. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{si } (x,y) \neq (0,0) \\ 0, & \text{si } (x,y) = (0,0) \end{cases}$

b. $f(x,y) = \begin{cases} \dfrac{x\cos\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} - \dfrac{\operatorname{sen}^2 x}{x^2 + y^2}, & \text{si } (x,y) \neq (0,0) \\ 0, & \text{si } (x,y) = (0,0) \end{cases}$

SOLUCION:

a. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{si } (x,y) \neq (0,0) \\ 0, & \text{si } (x,y) = (0,0) \end{cases}$

$\lim\limits_{x\to P_0} f(\bar{x}) = f(P_0)$

i) $f(P_0) = f(0,0) = 0 \ \exists$

ii) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2 + y^2}} = \dfrac{0}{0}$ indeterminado

Entonces nos definimos las curvas:
$C_1 = \{(x,y) \in R^2 / y = 0\}$ y $C_2 = \{(x,y) \in R^2 / y = x\}$

1. $\lim\limits_{\substack{x\to 0\\ x\in C_1}} f(\bar{x}) = \lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y)\in C_1}} \dfrac{xy}{\sqrt{x^2 + y^2}} = \lim\limits_{x\to 0} \dfrac{0}{\sqrt{x^2}} = 0 = L_1$

2. $\lim\limits_{\substack{x\to 0\\ x\in C_2}} f(\bar{x}) = \lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y)\in C_2}} \dfrac{xy}{\sqrt{x^2 + y^2}} = \lim\limits_{y\to 0} \dfrac{y^4}{\sqrt{y^4 + 1}} = 0 = L_2$

Como $L_1 = L_2$, entonces definimos $C_3 = \{(x,y) \in R^2 / y = \tan x\}$

$$\lim_{\substack{x\to 0\\ x\in C_3}} f(x)=\lim_{\substack{(x,y)\to(0,0)\\ (x,y)\in C_3}} \frac{xy}{\sqrt{x^2+y^2}}=\lim_{x\to 0}\frac{x\tan x}{\sqrt{x^2+\tan^2 x}}=\lim_{x\to 0}\frac{x\tan x}{\sqrt{x^2\left(1+\frac{\tan^2 x}{x^2}\right)}}$$

$$=\lim_{x\to 0}\frac{x\tan x}{x\sqrt{1+\frac{\tan^2 x}{x^2}}}=\frac{0}{\sqrt{1+1}}=0=L_3$$

Como $L_1=L_2=L_3$**, entonces** $\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}=0$

Por lo tanto $f(x,y)$ **es continua.**

b. $f(x,y)=\begin{cases}\dfrac{x\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}-\dfrac{sen^2 x}{x^2+y^2}, & si\ (x,y)\neq(0,0)\\ 0, & si\ (x,y)=(0,0)\end{cases}$

1. **En** $(x,y)\neq(0,0)$
   Sea $P_0=(x_0,y_0)\neq(0,0)$

$$\lim_{(x,y)\to(x_0,y_0)} P_0=\lim_{(x,y)\to(x_0,y_0)}\left(\frac{x\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}-\frac{sen^2 x}{x^2+y^2}\right)$$

$$=\frac{x_0\cos\sqrt{x_0^2+y_0^2}}{\sqrt{x_0^2+y_0^2}}-\frac{sen^2 x_0}{x_0^2+y_0^2}=f(x_0,y_0)$$

$f(x,y)$ **es continua en** $(x_0,y_0)\neq(0,0)$ **por lo tanto es continua en** $R^2-\{(0,0)\}$

2. **En** $(x,y)=(0,0)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x)=\lim_{(x,y)\to(x_0,y_0)}\left(\frac{x\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}-\frac{sen^2 x}{x^2+y^2}\right)=\infty-\infty \text{ indeterminado}$$

Entonces nos definimos las curvas:

$C_1=\{(x,y)\in R^2\ /\ y=0\}$ **y** $C_2=\{(x,y)\in R^2\ /\ y=x\}$

$$\lim_{\substack{x\to 0\\ x\in C_1}} f(x)=\lim_{\substack{(x,y)\to(0,0)\\ (x,y)\in C_1}}\left(\frac{x\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}-\frac{sen^2 x}{x^2+y^2}\right)=\lim_{x\to 0}\frac{x\cos\sqrt{x^2}}{\sqrt{x^2}}-\frac{sen^2 x}{x^2}$$

$$=\lim_{x\to 0}\frac{x\cos|x|}{|x|}-\frac{sen^2 x}{x^2}$$

Por definición de valor absoluto se sabe que: $|x|=\begin{cases}x & ;x\geq 0\\ -x & ;x<0\end{cases}$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{x\cos(-x)}{(-x)}-\left(\frac{senx}{x}\right)^2=\lim_{x\to 0^-}\frac{x\cos x}{-x}-\left(\frac{senx}{x}\right)^2=-2=L_1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x\cos x}{x}-\left(\frac{senx}{x}\right)^2=1-1=0=L_2$$

Como $L_1\neq L_2$**, entonces** $\lim_{x\to 0} f(x)$ **no existe**

Por lo tanto $\lim_{x\to 0} f(x)$ no existe, entonces no es continua en $(0,0)$

Ejercicios propuestos:

Probar los siguientes límites:

1. $\lim_{(x,y)\to(2,1)} (x^2 - 3y) = 1$ **Rpta:** $\delta = \min\left\{1, \frac{\varepsilon}{8}\right\}$
2. $\lim_{(x,y)\to(3,2)} (3x - 4y) = 1$ **Rpta:** $\delta = \frac{\varepsilon}{7}$
3. $\lim_{(x,y)\to(1,2)} (y^2 + x) = 5$ **Rpta:** $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$
4. $\lim_{(x,y)\to(-2,-2)} (3x^2 - 4y^2) = -4$ **Rpta:** $\delta = \min\left\{1, \frac{\varepsilon}{35}\right\}$

Encontrar los siguientes límites:

5. $\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$ **Rpta: No existe**
6. $\lim_{(x,y)\to(3,5)} (x^2 + xy - y)$ **Rpta: 19**
7. $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ **Rpta: No existe**
8. $\lim_{(x,y)\to(2,0)} \frac{x \tan y}{y}$ **Rpta: 2**
9. $\lim_{(x,y)\to(0,0)} \frac{x^2 + y}{x^2 + y^2}$ **Rpta: No existe**

DERIVADA DE FUNCIONES REALES DE VARIAS VARIABLES:

Definición:

Sea $f : S \subset R^2 \to R$ una función de dos variables independientes tal que $z = f(x,y)$, es decir:

$$f : R^2 \to R$$
$$(x,y) \to f(x,y) = z$$

a) La derivada parcial de la función f con respecto a la variable x está dado por:

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x; y) - f(x,y)}{\Delta x}; \ y \text{ es constante}$$

b) La derivada parcial de la función f con respecto a la variable y está dado por:

$$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y\to 0} \frac{f(x; y+\Delta y) - f(x,y)}{\Delta y}; \ x \text{ es constante}$$

Definición:

Sea $f : S \subset R^3 \to R$ una función real de tres variables independientes donde $w = f(x,y,z)$, es decir:

$$f : S \subset R^3 \to R$$

$$(x, y, z) \to f(x, y, z) = w$$

a) La derivada parcial de la función f con respecto a la variable x está dado por:

$$\frac{\partial w}{\partial x} = \frac{\partial f(x, y, z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x; y, z) - f(x, y, z)}{\Delta x}; \quad y, z \text{ constantes}$$

b) La derivada parcial de la función f con respecto a la variable y está dado por:

$$\frac{\partial w}{\partial y} = \frac{\partial f(x, y, z)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x; y + \Delta y, z) - f(x, y, z)}{\Delta y}; \quad x, z \text{ constantes}$$

c) La derivada parcial de la función f con respecto a la variable z está dado por:

$$\frac{\partial w}{\partial z} = \frac{\partial f(x, y, z)}{\partial z} = \lim_{\Delta z \to 0} \frac{f(x; y, z + \Delta z) - f(x, y, z)}{\Delta z}; \quad x, y \text{ constantes}$$

EJEMPLOS:

1. **Hallar las derivadas parciales de las siguientes funciones.**
   - **a.** $z = x^5 + y^5 - 3x^4y^4 + 3xy$
   - **b.** $z = x^{y^2}$
   - **c.** $z = \arctan\sqrt{\dfrac{x^2 - y^2}{x^2 + y^2}}$

SOLUCIÓN:

a) $z = x^5 + y^5 - 3x^4y^4 + 3xy$

$$\frac{\partial z}{\partial x} = 5x^4 - 12x^3y^4 + 3y \qquad \frac{\partial z}{\partial x} = 5x^4 - 12x^3y^4 + 3$$

b) $z = x^{y^2}$

$$\frac{\partial z}{\partial x} = y^2 x^{y^2 - 1} \qquad \frac{\partial z}{\partial x} = 2yx^{y^2} \ln x$$

c) $z = \arctan\sqrt{\dfrac{x^2 - y^2}{x^2 + y^2}}$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \left(\sqrt{\dfrac{x^2 - y^2}{x^2 + y^2}}\right)^2} \cdot \frac{\partial}{\partial x}\left(\sqrt{\frac{x^2 - y^2}{x^2 + y^2}}\right) = \frac{1}{1 + \dfrac{x^2 - y^2}{x^2 + y^2}} \cdot \frac{1}{2}\left(\frac{x^2 - y^2}{x^2 + y^2}\right)^{-1/2} \cdot \frac{\partial}{\partial x}\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$$

$$= \frac{x^2 + y^2}{2x^2} \cdot \frac{1}{2}\sqrt{\frac{x^2 + y^2}{x^2 - y^2}} \cdot \frac{2x(x^2 + y^2) - 2x(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$= \frac{1}{2x^2}\sqrt{\frac{x^2 + y^2}{x^2 - y^2}} \cdot \frac{2xy^2}{x^2 + y^2} = \frac{y^2}{x\sqrt{x^4 - y^4}}$$

$$\frac{\partial z}{\partial y}=\frac{1}{1+\left(\sqrt{\frac{x^2-y^2}{x^2+y^2}}\right)^2}.\frac{\partial}{\partial y}\left(\sqrt{\frac{x^2-y^2}{x^2+y^2}}\right)=\frac{1}{1+\frac{x^2-y^2}{x^2+y^2}}.\frac{1}{2}\left(\frac{x^2-y^2}{x^2+y^2}\right)^{-1/2}.\frac{\partial}{\partial y}\left(\frac{x^2-y^2}{x^2+y^2}\right)$$

$$=\frac{x^2+y^2}{2x^2}.\frac{1}{2}\sqrt{\frac{x^2+y^2}{x^2-y^2}}.\frac{(-2y)(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$=-\frac{1}{2x^2}\sqrt{\frac{x^2+y^2}{x^2-y^2}}.\frac{2x^2y}{x^2+y^2}=-\frac{y}{\sqrt{x^4-y^4}}$$

2. **Calcular** $\frac{\partial z}{\partial x}$ **y** $\frac{\partial z}{\partial y}$ **en** $z=e^{sen(y/x)}$

SOLUCIÓN:

$$\frac{\partial z}{\partial x}=e^{sen(y/x)}.\frac{\partial}{\partial x}\left(sen\frac{y}{x}\right)=e^{sen(y/x)}.cos\frac{y}{x}.\frac{\partial}{\partial x}\left(\frac{y}{x}\right)=e^{sen(y/x)}.cos\frac{y}{x}.\left(\frac{-y}{x^2}\right)$$

$$=\frac{-y}{x}.e^{sen(y/x)}.cos\frac{y}{x}$$

$$\frac{\partial z}{\partial y}=e^{sen(y/x)}.\frac{\partial}{\partial y}\left(sen\frac{y}{x}\right)=e^{sen(y/x)}.cos\frac{y}{x}.\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=e^{sen(y/x)}.cos\frac{y}{x}.\left(\frac{1}{x}\right)$$

$$=\frac{1}{x}.e^{sen(y/x)}.cos\frac{y}{x}$$

3. **Hallar las derivadas parciales de** $w=x^{x^2+y^2+z^2}$

SOLUCIÓN:

Sea $u=x$ **;** $v=x^2+y^2+z^2$ **, entonces** $w=u^v$
Utilizando formula se tiene:

$$\frac{\partial w}{\partial x}=x^{x^2+y^2+z^2}.\ln x.(2x)+x^{x^2+y^2+z^2-1}.x^2+y^2+z^2$$

$$\frac{\partial w}{\partial y}=x^{x^2+y^2+z^2}.\ln x.\frac{\partial}{\partial y}(x^2+y^2+z^2)=2y\ln x.x^{x^2+y^2+z^2}$$

$$\frac{\partial w}{\partial z}=x^{x^2+y^2+z^2}.\ln x.\frac{\partial}{\partial z}(x^2+y^2+z^2)=2z\ln x.x^{x^2+y^2+z^2}$$

4. **Si** $z=xy+xe^{y/x}$**; hallar** $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$

SOLUCIÓN:

$$\frac{\partial z}{\partial x}=y+x\frac{\partial}{\partial x}\left(e^{y/x}\right)+e^{y/x}\frac{\partial x}{\partial x}=y+xe^{y/x}.\frac{\partial}{\partial x}\left(\frac{y}{x}\right)+e^{y/x}$$

$$=y+xe^{y/x}\left(\frac{-y}{x^2}\right)+e^{y/x}\ =y-\frac{y}{x}e^{y/x}+e^{y/x}$$

$$\frac{\partial z}{\partial y}=x+xe^{y/x}.\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=x+xe^{y/x}.\frac{1}{x}=x+e^{y/x}$$

Luego:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left(y-\frac{y}{x}e^{y/x}+e^{y/x}\right)+y\left(x+e^{y/x}\right)=xy+xe^{y/x}+xy=z+xy$$

5. **Si** $u=\frac{e^{xyz}}{e^x+e^y+e^z}$ **; hallar** $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$

SOLUCIÓN:

$$\frac{\partial u}{\partial x}=\frac{\left(e^x+e^y+e^z\right)e^{xyz}.yz-e^{xyz}.e^x}{\left(e^x+e^y+e^z\right)^2}=\frac{e^{xyz}\left[\left(e^x+e^y+e^z\right)yz-e^x\right]}{\left(e^x+e^y+e^z\right)\left(e^x+e^y+e^z\right)}$$

$$=\frac{z\left[\left(e^x+e^y+e^z\right)yz-e^x\right]}{e^x+e^y+e^z}$$

$$\frac{\partial u}{\partial y}=\frac{\left(e^x+e^y+e^z\right)e^{xyz}.xz-e^{xyz}.e^y}{\left(e^x+e^y+e^z\right)^2}=\frac{e^{xyz}\left[\left(e^x+e^y+e^z\right)xz-e^y\right]}{\left(e^x+e^y+e^z\right)\left(e^x+e^y+e^z\right)}$$

$$=\frac{z\left[\left(e^x+e^y+e^z\right)xz-e^y\right]}{e^x+e^y+e^z}$$

$$\frac{\partial u}{\partial y}=\frac{\left(e^x+e^y+e^z\right)e^{xyz}.xy-e^{xyz}.e^z}{\left(e^x+e^y+e^z\right)^2}=\frac{e^{xyz}\left[\left(e^x+e^y+e^z\right)xy-e^z\right]}{\left(e^x+e^y+e^z\right)\left(e^x+e^y+e^z\right)}$$

$$=\frac{z\left[\left(e^x+e^y+e^z\right)xy-e^z\right]}{e^x+e^y+e^z}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{z}{e^x+e^y+e^z}\left[\left(e^x+e^y+e^z\right)\left(xy+xz+yz\right)+\left(e^x+e^y+e^z\right)\right]$$

$$=z\left(xy+xz+yz+1\right)$$

6. **Si** $u=\frac{z}{\sqrt[3]{x^2+y^2}}$**, hallar** $3\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$

SOLUCIÓN:

$$\frac{\partial u}{\partial x}=\frac{\sqrt[3]{x^2+y^2}\,(0)-z.\frac{1}{3}\left(x^2+y^2\right)^{-2/3}.2x}{\left(\sqrt[3]{x^2+y^2}\right)^2}=\frac{\frac{-2xz}{3}}{\left(x^2+y^2\right)^{2/3}.\left(x^2+y^2\right)^{2/3}}=\frac{-2xz}{3\left(x^2+y^2\right)^{4/3}}$$

$$\frac{\partial u}{\partial y}=\frac{\sqrt[3]{x^2+y^2}\,(0)-z.\frac{1}{3}\left(x^2+y^2\right)^{-2/3}.2y}{\left(\sqrt[3]{x^2+y^2}\right)^2}=\frac{-2yz}{3\left(x^2+y^2\right)^{4/3}}$$

$$\frac{\partial u}{\partial z}=\frac{1}{\sqrt[3]{x^2+y^2}}$$

$$3\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=3\left[\frac{-2x^2z}{3(x^2+y^2)^{4/3}}-\frac{2y^2z}{3(x^2+y^2)^{4/3}}+\frac{z}{\sqrt[3]{x^2+y^2}}\right]$$

$$=3\left[\frac{-2z(x^2+y^2)}{3(x^2+y^2)^{4/3}}+\frac{z}{\sqrt[3]{x^2+y^2}}\right]$$

$$=\left[\frac{-2z}{\sqrt[3]{x^2+y^2}}+\frac{3z}{\sqrt[3]{x^2+y^2}}\right]=\frac{z}{\sqrt[3]{x^2+y^2}}=u$$

Ejercicios propuestos**:**

Determinar las derivadas parciales de las siguientes funciones:

1. $f(x,y)=x^3y+e^{xy}$ **Rpta:** $\begin{cases} f_x=3x^2y+y^2e^{xy^2} \\ f_y=x^3+2xye^{xy^2} \end{cases}$

2. $f(x,y)=\arctan\sqrt{\dfrac{x^2-y^2}{x^2+y^2}}$ **Rpta:** $\begin{cases} f_x=\dfrac{y^2}{x}(x^4-y^4)^{-1/2} \\ f_y=-y(x^4-y^4)^{-1/2} \end{cases}$

3. $f(x,y)=e^{y/x}\ln\dfrac{x^2}{y}$ **Rpta:** $\begin{cases} f_x=\dfrac{1}{x^2}e^{y/x}\left(2x-y\ln\dfrac{x^2}{y}\right) \\ f_y=\dfrac{1}{xy}e^{y/x}\left(y\ln\dfrac{x^2}{y}-x\right) \end{cases}$

4. $z=\sqrt{x^2-y^2}\tan\dfrac{z}{\sqrt{x^2-y^2}}$ **Rpta:** $\begin{cases} z_x=\dfrac{xz}{x^2-y^2} \\ z_y=-\dfrac{yz}{x^2-y^2} \end{cases}$

Verificar si las siguientes funciones satisfacen las ecuaciones dadas.

5. $f(x,y)=\dfrac{x^2y^2}{x^2+y^2};xf_x+yf_y=2f$
6. $f(x,y)=\dfrac{Ax^n+By^n}{Cx^2+Dy^2};xf_x+yf_y=(n-2)f$
7. $f(x,y)=\sqrt{xy+\arctan(x/y)};xff_x+yff_y=xy$
8. $f(x,y)=\sqrt{x^2+y^2}\arctan\dfrac{y}{x};xf_x+yf_y=f$

DERIVACIÓN IMPLICITA

Definición:

Sea $f:S\subset R^2\to R$ una función real de dos variables independientes, se dice que la ecuación $E(x,y)=0$ define a "y" implícitamente como función de "x"; es decir $y=f(x)$; $\forall x$

Para hallar las derivadas parciales de funciones implícitas de varias variables se sigue lo siguiente:

1. $E(x, y, z) = 0$ **una función real de tres variables define implícitamente a** $z = f(x, y)$

$$\frac{\partial z}{\partial x} = -\frac{E_x}{E_z} \quad ; \quad \frac{\partial z}{\partial y} = -\frac{E_y}{E_z}$$

2. $E(x, y, z, w) = 0$ **una función real de cuatro variables, define implícitamente a** $w = f(x, y, z)$

$$\frac{\partial w}{\partial x} = -\frac{E_x}{E_w} \quad ; \quad \frac{\partial w}{\partial y} = -\frac{E_y}{E_w} \quad ; \quad \frac{\partial w}{\partial z} = -\frac{E_z}{E_w}$$

<u>EJEMPLOS:</u>

1. **Suponiendo que** w **es función de todas las otras variables, hallar las derivadas correspondientes de las siguientes funciones.**

a) $w - e^{wsen(y/x)} = 1$
b) $w - (r^2 + s^2)\cosh(rw) = 0$

SOLUCIÓN**:**

a) $w - e^{wsen(y/x)} = 1 \Rightarrow E(x, y, w) = w - e^{wsen(y/x)} - 1 = 0$
$\Rightarrow w = f(x, y)$

$$E_x = \frac{\partial E}{\partial x} = -e^{wsen(y/x)}.\frac{\partial}{\partial x}(wsen(y/x)) = -e^{wsen(y/x)}.w\cos(y/x)\frac{\partial}{\partial x}(y/x)$$

$$= -e^{wsen(y/x)}.w\cos(y/x).(-y/x^2) = \frac{wy}{x^2}\cos\left(\frac{y}{x}\right).e^{wsen(y/x)}$$

$$E_y = \frac{\partial E}{\partial y} = -e^{wsen(y/x)}.\frac{\partial}{\partial y}(wsen(y/x)) = -e^{wsen(y/x)}.w\cos(y/x).\frac{1}{x}$$

$$= \frac{-w}{x}\cos\left(\frac{y}{x}\right).e^{wsen(y/x)}$$

$$E_w = \frac{\partial E}{\partial w} = 1 - e^{wsen(y/x)}.\frac{\partial}{\partial w}(wsen(y/x)) = 1 - sen(y/x).e^{sen(y/x)}$$

Por lo tanto:

$$\frac{\partial w}{\partial x} = \frac{-E_x}{E_w} = \frac{\frac{wy}{x^2}\cos\left(\frac{y}{x}\right).e^{wsen(y/x)}}{1 - sen(y/x).e^{sen(y/x)}}$$

$$\frac{\partial w}{\partial y} = \frac{-E_y}{E_w} = \frac{\frac{-w}{x}\cos\left(\frac{y}{x}\right).e^{wsen(y/x)}}{1 - sen(y/x).e^{sen(y/x)}}$$

b) $w - (r^2 + s^2)\cosh(rw) = 0 \Rightarrow E(r, s, w) = w - (r^2 + s^2)\cosh(rw)$
$\Rightarrow w = f(x, y)$

$$E_r = \frac{\partial E}{\partial r} = -\left[\left(r^2+s^2\right)\mathrm{senh}(rw).w + \cosh(rw).2r\right] = 2r\cosh(rw) - \left(r^2+s^2\right)\mathrm{senh}(rw)$$

$$E_s = \frac{\partial E}{\partial s} = -2s\cosh(rw)$$

$$E_w = \frac{\partial E}{\partial w} = 1 - \left(r^2+s^2\right)\mathrm{senh}(rw).r = 1 - r\left(r^2+s^2\right)\mathrm{senh}(rw)$$

Por lo tanto:

$$\frac{\partial w}{\partial r} = \frac{-E_r}{E_w} = \frac{-2r\cosh(rw) + \left(r^2+s^2\right)\mathrm{senh}(rw)}{1 - r\left(r^2+s^2\right)\mathrm{senh}(rw)}$$

$$\frac{\partial w}{\partial s} = \frac{-E_s}{E_w} = \frac{2s\cosh(rw)}{1 - r\left(r^2+s^2\right)\mathrm{senh}(rw)}$$

2. **Hallar las derivadas respectivas de:**

   a) $x\cos y + y\cos z + z\cos x = 1$; $z = f(x,y)$

   b) $z = \sqrt{x^2-y^2}\tan\dfrac{z}{\sqrt{x^2-y^2}}$; $z = f(x,y)$

SOLUCIÓN:

a) $E(x,y,z) = x\cos y + y\cos z + z\cos x - 1 = 0$

$$E_x = \frac{\partial E}{\partial x} = \cos y - z\,\mathrm{sen}x$$

$$E_y = \frac{\partial E}{\partial y} = -x\,\mathrm{sen}y + \cos z$$

$$E_z = \frac{\partial E}{\partial z} = -y\,\mathrm{sen}z + \cos x$$

Por lo tanto:

$$\frac{\partial z}{\partial x} = \frac{-E_x}{E_z} = \frac{-\cos y + z\,\mathrm{sen}x}{-y\,\mathrm{sen}z + \cos x}$$

$$\frac{\partial z}{\partial y} = \frac{-E_y}{E_z} = \frac{x\,\mathrm{sen}y - \cos z}{-y\,\mathrm{sen}z + \cos x}$$

b) $z = \sqrt{x^2-y^2}\tan\dfrac{z}{\sqrt{x^2-y^2}} \Rightarrow \dfrac{z}{\sqrt{x^2-y^2}} - \tan\dfrac{z}{\sqrt{x^2-y^2}} = 0$

$$E(x,y,z) = \frac{z}{\sqrt{x^2-y^2}} - \tan\frac{z}{\sqrt{x^2-y^2}} = 0$$

$$\Rightarrow \ E(x,y,z) = z\left(x^2-y^2\right)^{-1/2} - \tan z\left(x^2-y^2\right)^{-1/2} = 0$$

$$E_x=\frac{\partial E}{\partial x}=z.\left(\frac{-1}{2}\right)\left(x^2-y^2\right)^{-3/2}.2x-\sec^2 z\left(x^2-y^2\right)^{-1/2}.\frac{\partial}{\partial x}\left[z\left(x^2-y^2\right)^{-1/2}\right]$$

$$=\frac{-zx}{\left(x^2-y^2\right)^{3/2}}+\frac{zx}{\left(x^2-y^2\right)^{3/2}}.\sec^2 z\left(x^2-y^2\right)^{-1/2}$$

$$=\frac{zx}{\left(x^2-y^2\right)^{3/2}}\left(\sec^2 z\left(x^2-y^2\right)^{-1/2}-1\right)=\frac{zx}{\left(x^2-y^2\right)^{3/2}}\tan^2 z\left(x^2-y^2\right)^{-1/2}$$

$$=\frac{zx}{\left(x^2-y^2\right)^{3/2}}\tan^2\frac{z}{\sqrt{x^2-y^2}}$$

$$E_y=\frac{\partial E}{\partial y}=z.\left(\frac{-1}{2}\right)\left(x^2-y^2\right)^{-3/2}.(-2y)-\sec^2 z\left(x^2-y^2\right)^{-1/2}.z.\frac{1}{2}\left(x^2-y^2\right)^{-3/2}(2y)$$

$$=\frac{zy}{\left(x^2-y^2\right)^{3/2}}-\sec^2 z\left(x^2-y^2\right)^{-1/2}.\frac{zy}{\left(x^2-y^2\right)^{3/2}}$$

$$=\frac{-zy}{\left(x^2-y^2\right)^{3/2}}.\tan^2 z\left(x^2-y^2\right)^{-1/2}=\frac{-zy}{\left(x^2-y^2\right)^{3/2}}.\tan^2\frac{z}{\sqrt{x^2-y^2}}$$

$$E_z=\frac{\partial E}{\partial z}=\left(x^2-y^2\right)^{-1/2}-\sec^2 z\left(x^2-y^2\right)^{-1/2}.\left(x^2-y^2\right)^{-1/2}$$

$$=\frac{-1}{\sqrt{x^2-y^2}}\left(\sec^2 z\left(x^2-y^2\right)^{-1/2}-1\right)=\frac{-1}{\sqrt{x^2-y^2}}.\tan^2\frac{z}{\sqrt{x^2-y^2}}$$

Por lo tanto:

$$\frac{\partial z}{\partial x}=-\frac{E_x}{E_z}=\frac{\dfrac{zx}{\left(x^2-y^2\right)^{3/2}}\tan^2\dfrac{z}{\sqrt{x^2-y^2}}}{\dfrac{1}{\sqrt{x^2-y^2}}.\tan^2\dfrac{z}{\sqrt{x^2-y^2}}}=\frac{zx}{x^2-y^2}$$

$$\frac{\partial z}{\partial y}=-\frac{E_y}{E_z}=\frac{\dfrac{zy}{\left(x^2-y^2\right)^{3/2}}.\tan^2\dfrac{z}{\sqrt{x^2-y^2}}}{\dfrac{-1}{\sqrt{x^2-y^2}}.\tan^2\dfrac{z}{\sqrt{x^2-y^2}}}=\frac{-zy}{x^2-y^2}$$

3. **Si u y v son funciones de x e y definidas implícitamente para las ecuaciones** $3x+y=u^2-v$; $x-2y=u-2v^2$ **hallar** $\frac{\partial u}{\partial x}$ **y** $\frac{\partial v}{\partial y}$

SOLUCIÓN**:**

a) **Derivando ambas ecuaciones respecto a la variable x:**

$$3=2u\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}\text{..........(1)}$$

$$1=\frac{\partial u}{\partial x}-4v\frac{\partial v}{\partial x}\text{...........(2)}$$

Multiplicando por $(-4v)$ **a la primera ecuación:**

$$-12v=-8uv\frac{\partial u}{\partial x}+4v\frac{\partial v}{\partial x}\ \ldots\ldots\ldots\ldots(1^*)$$

Sumando miembro a miembro las ecuaciones (2) y (1*)

$$-12v+1=(1-8uv)\frac{\partial u}{\partial x}\Rightarrow\frac{\partial u}{\partial x}=\frac{1-12v}{1-8uv}$$

b) **Derivando ambas ecuaciones respecto a la variable y:**

$$1=2u\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}\ \ldots\ldots\ldots(1)$$

$$-2=\frac{\partial u}{\partial y}-4v\frac{\partial v}{\partial y}\ \ldots\ldots\ldots\ldots(2)$$

Multiplicando por $(-2u)$ a la segunda ecuación:

$$-4u=-2u\frac{\partial u}{\partial y}+8uv\frac{\partial v}{\partial y}\ \ldots\ldots\ldots\ldots(2^*)$$

Sumando miembro a miembro las ecuaciones (1) y (2*)

$$1-4u=(8uv-1)\frac{\partial v}{\partial y}\Rightarrow\frac{\partial v}{\partial y}=\frac{1-4u}{8uv-1}$$

4. **Si $u=(x+z)e^{y+z}\ \wedge\ z=f(x,y)$. Hallar $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}$**

SOLUCIÓN:

$$\frac{\partial u}{\partial x}=(x+z)\frac{\partial}{\partial x}\left(e^{y+z}\right)+e^{y+z}\frac{\partial}{\partial x}(x+z)$$

$$=(x+z)e^{y+z}.\frac{\partial z}{\partial x}+e^{y+z}\left(1+\frac{\partial z}{\partial x}\right)=e^{y+z}\left[1+\frac{\partial z}{\partial x}+(x+z)\frac{\partial z}{\partial x}\right]$$

$$=e^{y+z}\left[1+(1+x+z)\frac{\partial z}{\partial x}\right]$$

$$\frac{\partial u}{\partial y}=(x+z)\frac{\partial}{\partial x}\left(e^{y+z}\right)+e^{y+z}\frac{\partial}{\partial x}(x+z)$$

$$=(x+z)e^{y+z}\left(1+\frac{\partial z}{\partial y}\right)+e^{y+z}\frac{\partial z}{\partial y}$$

$$=e^{y+z}\left[x+z+(1+x+z)\frac{\partial z}{\partial y}\right]$$

Entonces:

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=e^{y+z}\left[(1+x+z)+(1+x+z)\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)\right]$$

$$=e^{y+z}(1+x+z)\left(1+\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)$$

5. **Si $u=xyz\ \wedge\ z=f(x,y)$ demostrar que:**

$$z\left(x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)=u\left(x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}\right)$$

SOLUCIÓN:

$$\frac{\partial u}{\partial x} = xy\frac{\partial z}{\partial x} + z\frac{\partial}{\partial x}(xy) = xy\frac{\partial z}{\partial x} + z\left(x\frac{\partial y}{\partial x} + y\frac{\partial x}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = xy\frac{\partial z}{\partial x} + zy$$

$$\frac{\partial u}{\partial y} = xy\frac{\partial z}{\partial y} + z\frac{\partial}{\partial y}(xy) = xy\frac{\partial z}{\partial y} + z\left(x\frac{\partial y}{\partial y} + y\frac{\partial x}{\partial y}\right)$$

$$\frac{\partial u}{\partial y} = xy\frac{\partial z}{\partial y} + zx$$

Luego:

$$z\left[\left(x^2y\frac{\partial z}{\partial x} + xzy\right) - \left(xy^2\frac{\partial z}{\partial y} + zxy\right)\right] = xyz\left(x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right)$$

$$= u\left(x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right)$$

Ejercicios propuestos:

Determinar las derivadas parciales de las siguientes funciones:

1. $f(x,y,z) = 2x^3 + 4y^2 - 2xy\cos z$ **Rpta:** $\begin{cases} f_x = 6x^2 - 2y\cos z \\ f_y = 8y - 2x\cos z \\ f_z = 2xysenz \end{cases}$

2. $f(x,y,z) = \dfrac{x^2 - y^2}{y^2 + z^2}$ **Rpta:** $\begin{cases} f_x = 2x\left(y^2 + z^2\right)^{-1} \\ f_y = -2y\left(x^2 + z^2\right)\left(y^2 + z^2\right)^{-2} \\ f_z = -2z\left(x^2 - z^2\right)\left(y^2 + z^2\right)^{-2} \end{cases}$

En los siguientes ejercicios demostrar lo que se indica:

3. $f(x,y,z) = \left(\dfrac{x - y + z}{x + y - z}\right)^n ; xf_x + yf_y + zf = 0_z$
4. $z^2 + \dfrac{2}{x} = \sqrt{y^2 - z^2} ; x^2\dfrac{\partial z}{\partial x} + \dfrac{1}{y}\dfrac{\partial z}{\partial y} = \dfrac{1}{z}$
5. $f(x,y,z) = xy^2 + y^2z + z^2x;\ f_x + f_y + f_z = (x + y + z)^2$

DERIVADAS PARCIALES DE ORDEN SUPERIOR

Sea $f : S \subset R^2 \to R$ una función definida en el conjunto abierto S talque:

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}$$

Son sus derivadas parciales de primer orden de la función f con respecto a las variables x e y entonces las funciones definidas por:

$$f_{xx}(x,y) = \lim_{\Delta x \to 0} \frac{f_x(x+\Delta x, y) - f_x(x,y)}{\Delta x}$$
$$f_{yy}(x,y) = \lim_{\Delta y \to 0} \frac{f_y(x, y+\Delta y) - f_y(x,y)}{\Delta y}$$
$$f_{yx}(x,y) = \lim_{\Delta x \to 0} \frac{f_y(x+\Delta x, y) - f_y(x,y)}{\Delta x}$$
$$f_{xy}(x,y) = \lim_{\Delta y \to 0} \frac{f_y(x, y+\Delta y) - f_y(x,y)}{\Delta y}$$

Si los límites existen son llamadas derivadas parciales de segundo orden de la función f , también se denotan por:

$$\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial^2 f(x,y)}{\partial x^2} = f_{xx}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial y}\right) = \frac{\partial^2 f(x,y)}{\partial y^2} = f_{yy}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = \frac{\partial^2 f(x,y)}{\partial x \partial y} = f_{yx}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial^2 f(x,y)}{\partial y \partial x} = f_{xy}$$

Siguiendo el mismo procedimiento podemos hallar las derivadas parciales de orden más superior, como por ejemplo.

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right) = \frac{\partial^3 f(x,y)}{\partial x \partial x \partial y} = f_{yxx}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial^3 f(x,y)}{\partial x \partial x \partial y}\right) = \frac{\partial^4 f(x,y)}{\partial y \partial x \partial x \partial y} = f_{yxxy}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial^4 f(x,y)}{\partial y \partial x \partial x \partial y}\right) = \frac{\partial^5 f(x,y)}{\partial y \partial y \partial x \partial x \partial y} = f_{yxxyy}$$

NOTA**:**

La notación con operadores indica que el orden de derivación es de derecha a izquierda, mientras la notación con sub índices indican que el orden de derivación es de izquierda a derecha.

PROPOSICIÓN**:**

Sea $f : S \subset R^2 \to R$ una función definida en el conjunto abierto S tal que la función y sus derivadas parciales de orden superior son funciones continuas en el conjunto abierto S

entonces se cumple que sus derivadas parciales mixtas de segundo orden son iguales para todo par $(x,y)\in S$, es decir:

$$f_{xy}(x,y)=f_{yx}(x,y);\forall(x,y)\in S$$

Una consecuencia importante de esta proposición es que en una función de varias variables se puede cambiar el orden de derivación sin que por ello se altere el resultado siempre en cuando la función y las derivadas parciales de orden superior sean funciones continuas. Por ejemplo la función f y sus derivadas parciales de orden superior son funciones continuas, entonces se tiene que:

$$f_{xxxyyy}=f_{xyxyxy}=f_{xxyyxy}=f_{yyyxxx}=\text{.........}$$

EJEMPLOS**:**

1. **Hallar las derivadas parciales de segundo orden de las siguientes funciones:**
   a. $f(x,y)=x^4+y^4-4xy^2$
   b. $f(x,y)=\arctan\left(\dfrac{x+y}{1-xy}\right)$
   c. $f(x,y,z)=\ln\left(\dfrac{1+x}{1+z}\right)-e^{xy}$
   d. $x^2+y^2+z^2=a^2;a>0$

   SOLUCIÓN

   a. $f(x,y)=x^4+y^4-4xy^2$

   $f_x(x,y)=4x^3+y^4-4xy^2$

   $f_{xx}(x,y)=12x^2$

   $f_y(x,y)=4y^3-8xy$

   $f_{yy}(x,y)=12y^2-8x$

   $f_{xy}(x,y)=-8y$

   $f_{yx}(x,y)=-8y$

   b. $f(x,y)=\arctan\left(\dfrac{x+y}{1-xy}\right)$

$$f_x=\frac{1}{1+\left(\dfrac{x+y}{1-xy}\right)^2}\cdot\frac{d}{dx}\left(\frac{x+y}{1-xy}\right)$$

$$f_x=\frac{(1-xy)^2}{(1-xy)^2+(x+y)^2}\,\frac{(1-xy)-(x+y)(-y)}{(1-xy)^2}$$

$$f_x=\frac{1-xy+xy+y^2}{1-2xy+x^2y^2+x^2+2xy+y^2}$$

$$f_x=\frac{1+y^2}{(1+y^2)+x^2(1+y^2)}$$

$$f_x=\frac{(1-xy)^2}{(1-xy)^2(x+y)^2}=\frac{1}{1+x^2}$$

$$f_y = \frac{(1-xy)^2}{(1-xy)^2+(x+y)^2}\cdot\frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2}$$

$$f_y = \frac{1-xy+x^2+xy}{1-2xy+x^2y^2+x^2+2xy+y^2}$$

$$f_y = \frac{1+x^2}{(1+x^2)+y^2(1+y^2)} = \frac{1+x^2}{(1+x^2)(1+y^2)} = \frac{1}{1+y^2}$$

$$f_{xx} = \frac{-2x}{(1+x^2)^2}$$

$$f_{yy} = \frac{-2y}{(1+y^2)^2}$$

$$f_{xy} = f_{yx} = 0$$

c. $f(x,y,z) = \ln\left(\frac{1+x}{1+z}\right) - e^{xy}$

$$f_x = \frac{1}{\frac{1+x}{1+z}}\cdot\frac{d}{dx}\left(\frac{1+x}{1+z}\right) - e^{xy}\frac{d}{dx}(xy)$$

$$f_x = \frac{1+z}{1+x}\cdot\frac{(1+z)}{(1+z)^2} - e^{xy}y = \frac{1}{1+x} - ye^{xy}$$

$$f_y = -xe^{xy}$$

$$f_z = \frac{1+z}{1+x}\cdot\frac{-(1+x)}{(1+z)^2} = -\frac{1}{1+z}$$

$$f_{xx} = -\frac{1}{(1+x)^2} - y^2e^{xy}$$

$$f_{yx} = -\left(xye^{xy}+e^{xy}\right) = f_{xy}$$

$$f_{yy} = -\left(x^2e^{xy}\right)$$

$$f_{xz} = f_{yz} = 0$$

2. **Hallar las derivadas parciales del orden que se indica.**
   a. f_{xxy} **si** $f(x,y) = x\ln(xy)$
   b. f_{yyyxxx} **si** $f(x,y) = x^3\text{seny} + y^3\text{senx}$

SOLUCIÓN

a. $f_x = x.\frac{1}{xy}.y + \ln(xy) = 1 + \ln(xy)$

$$f_{xx} = \frac{y}{xy} = \frac{1}{x}$$

$$f_{xxy} = 0$$

b. $f_y = x^3\cos y + 3y^2\text{senx}$

$$f_{yy} = -x^3\text{seny} + 6y\,\text{senx}$$

$$f_{yyy} = -x^3 \cos y + 6 \operatorname{sen} x$$
$$f_{yyyx} = -3x^2 \cos y + 6 \cos x$$
$$f_{yyyxx} = -6x \cos y - 6 \operatorname{sen} x$$
$$f_{yyyxxx} = -6 \cos y - 6 \cos x = -6(\cos y + \cos x)$$

3. **Si** $z = e^x (x \cos y - y \operatorname{sen} y)$**, hallar:** $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$

SOLUCIÓN

$$\frac{\partial z}{\partial x} = e^x \cos y + (x \cos y - y \operatorname{sen} y) e^x$$
$$\frac{\partial^2 z}{\partial x^2} = e^x \cos y + e^x \cos y + e^x (x \cos y - y \operatorname{sen} y)$$
$$\frac{\partial z}{\partial y} = e^x (-x \operatorname{sen} y - y \cos y - \operatorname{sen} y) = e^x (-(1 + x) \operatorname{sen} y - y \cos y)$$
$$= -e^x ((1 + x) \operatorname{sen} y + y \cos y)$$
$$\frac{\partial^2 z}{\partial y^2} = e^x (-x \cos y - (-y \operatorname{sen} y + \cos y) - \cos y)$$
$$= e^x (-x \cos y + y \operatorname{sen} y - 2 \cos y)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left[e^x \cos y + e^x \cos y + e^x (x \cos y - y \operatorname{sen} y)\right]$$
$$+ \left[e^x (-x \cos y + y \operatorname{sen} y - 2 \cos y)\right]$$
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \cos y + e^x \cos y - 2e^x \cos y = 0$$

4. **Si** $z = \ln[(x + y)(x - y)]$**, hallar** $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial y \partial x} + \dfrac{\partial^2 z}{\partial y^2}$

SOLUCIÓN

$$\frac{\partial z}{\partial x} = \frac{1}{(x + y)(x - y)} . (x + y)(1) + (x - y)(1) = x + y + x - y = 2x$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{-2x(2x) + 2(x^2 - y^2)}{(x^2 - y^2)^2} = \frac{-4x^2 + 2x^2 + 2y^2}{(x^2 - y^2)^2} = \frac{-2(x^2 + y^2)}{(x^2 - y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{(x + y)(x - y)} . \frac{(x + y)(-1) + (x - y)(1)}{x^2 - y^2} = \frac{-2y}{x^2 - y^2}$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{-2(x^2 - y^2) - (-2y)(-2y)}{(x^2 - y^2)^2} = \frac{-2x^2 + 2y^2 - 4y^2}{(x^2 - y^2)^2} = \frac{-2(x^2 + y^2)}{(x^2 - y^2)^2}$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial z}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{-2x(-2y)}{(x^2 - y^2)^2} = \frac{4xy}{(x^2 - y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial^2 z}{\partial y^2} = \frac{-2(x^2+y^2) - 8xy - 2(x^2+y^2)}{(x^2+y^2)^2}$$

5. **Dada la función $f(x, y) = e^{ax+by} g(x, y)$ donde $g_x(x, y) = g_y(x, y) = 1$. Hallar los valores de las constantes a y b tales que: $f_x(x, y) = f_y(x, y)$; $1 + f_{xy}(x, y) = a + f_{yx}(x, y)$**

SOLUCIÓN:

$$f_x = e^{ax+by} \frac{\partial}{\partial x} g(x, y) + g(x, y) \frac{\partial}{\partial x}\left(e^{ax+by}\right)$$

$$f_x = e^{ax+by} \underbrace{g_x(x, y)}_{1} + ag(x, y)e^{ax+by}$$

$$f_x = e^{ax+by} + ag(x, y)e^{ax+by} = e^{ax+by}(1 + ag(x, y))$$

$$f_y = e^{ax+by} \frac{\partial}{\partial y} g(x, y) + g(x, y) \frac{\partial}{\partial y}\left(e^{ax+by}\right)$$

$$f_y = e^{ax+by} \underbrace{g_y(x, y)}_{1} + bg(x, y)e^{ax+by}$$

$$f_y = e^{ax+by} + bg(x, y)e^{ax+by} = e^{ax+by}(1 + bg(x, y))$$

Como $f_x(x, y) = f_y(x, y)$

$$e^{ax+by}(1 + ag(x, y)) = e^{ax+by}(1 + bg(x, y)) \Rightarrow a = b$$

$$f_{xy} = be^{ax+by} + a\left[g(x, y)be^{ax+by} + g_y(x, y)e^{ax+by}\right]$$

$$f_{xy} = e^{ax+by}(b + abg(x, y) + a)$$

$$f_{yx} = ae^{ax+by} + b\left[g(x, y)ae^{ax+by} + g_x(x, y)e^{ax+by}\right]$$

$$f_{yx} = e^{ax+by}(a + abg(x, y) + b)$$

Luego: $1 + f_{xy}(x, y) = a + f_{yx}(x, y)$

$$1 + e^{ax+by}(b + abg(x, y) + a) = a + e^{ax+by}(a + abg(x, y) + b)$$

$$\Rightarrow a = 1$$

Por lo tanto: $a = b = 1$

INCREMENTO Y DIFERENCIAL DE FUNCIONES DE VARIAS VARIABLES

Sea $f : S \subset R \to R$ una función definida en el conjunto abierto S tal que $y = f(x)$ sabemos que el incremento de la función f en el punto predeterminado a S esta dado por:

$$\Delta f(x) = f(x + \Delta x) - f(x)$$

También si $f: S \subset R^3 \to R$ es una función de tres variables independientes definida en S tal que $w = f(x, y, z)$ entonces el incremento de f en el punto $(x, y, z) \in S$ está dado por:

$$\Delta f(x, y, z) = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

Siguiendo el mismo procedimiento se puede hallar los incrementos de funciones de cualquier número de variable.

DEFINICIÓN:

Si $f: S \subset R^2 \to R$ una función de dos variables independientes definida en el punto $(x, y) \in S$ entonces la diferencial total de la función f es la función df definida por:

$$df = f_x(x, y)\Delta x + f_y(x, y)\Delta y$$

O lo que es lo mismo: $df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ **donde:** $\Delta y = dy$

Del mismo modo si $f: S \subset R^3 \to R$ es una función definida en S tal que $w = f(x, y, z)$ entonces la diferencial total de función f en el punto $(x, y, z) \in S$ está dado por:

$$df(x, y, z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Y así sucesivamente se puede hallar la diferencial total de funciones de cualquier número de variables.

OBSERVACIONES:

Se tiene que: $\Delta f(x, y) \cong f_x(x, y)\Delta x + f_y(x, y)\Delta y \cong \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \cong df(x, y)$

O lo que es lo mismo: $f(x + \Delta x, y + \Delta y) - f(x, y) \cong df(x, y)$

De aquí que el incremento de una función se puede aproximar por diferenciales.

Por otro lado si una cantidad $z = f(x, y)$ se aproxima mediante otra cantidad $f(x + \Delta x, y + \Delta y)$ con un error de $\Delta z = \Delta f$ se tiene los siguientes valores:

1. $\frac{\Delta z}{z} = \frac{\Delta f}{f}$ **se llama error relativo.**
2. $100\frac{\Delta z}{z} = 100\frac{\Delta f}{f}$ **se llama error porcentual**

Y teniendo en cuenta el hecho de que el incremento de la función $\Delta f \cong df$ se tiene:

1'. $\frac{dz}{z} = \frac{df}{f}$ **error relativo aproximado**

2'. $100\frac{dz}{z} = 100\frac{df}{f}$ **error porcentual aproximado**

DIFERENCIALES DE ORDEN SUPERIOR

Dado una función de dos variables independientes con derivadas parciales de orden superior continuas en el conjunto S sabemos que:

$$d^2 f = d(df) = dg$$

$$d^2 f = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy$$

$$\Rightarrow d^2 f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right)dy$$

$$= \left(\frac{\partial^2 f}{\partial x^2} dx + \frac{\partial^2 f}{\partial y \partial x} dy\right)dx + \left(\frac{\partial^2 f}{\partial x \partial y} dx + \frac{\partial^2 f}{\partial y^2} dy\right)dy$$

$$= \frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{\partial^2 f}{\partial y \partial x}(dydx) + \frac{\partial^2 f}{\partial x \partial y}(dxdy) + \frac{\partial^2 f}{\partial y^2}(dy)^2$$

Como f y sus derivadas de orden superior son continuas, entonces: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\frac{\partial^2 f}{\partial y \partial x} dxdy + \frac{\partial^2 f}{\partial y^2}(dy)^2$$

$$d^2 f = \left(\frac{\partial^2}{\partial x^2}(dx)^2 + 2\frac{\partial^2}{\partial y \partial x} dxdy + \frac{\partial^2}{\partial y^2}(dy)^2\right) f$$

$$d^2 f = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^2 f$$

Procediendo en forma similar la diferencial de orden "n" para la función f está dado por:

$$d^n f = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^n f$$

Análogamente es una función de 3 variables independientes $w = g(x, y, z)$ tenemos que:

$$dw = dg(x, y, z) = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz\right) g$$

Entonces:

$$d^n w = d^n g = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz\right)^n g$$

EJEMPLOS:

1**. Hallar la diferencial total de las siguientes funciones.**

- **a.** $f(x, y) = x^2 - xy + y^2$
- **b.** $f(x, y) = \frac{xy}{x^2 + y^2}$
- **c.** $f(x, y) = e^{xy}\cos(x + y)$
- **d.** $\mu(x, y, z) = (xy + \frac{x}{y})^z$

SOLUCIÓN

a. $f(x, y) = x^2 - xy + y^2$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$f_X = \frac{\partial f}{\partial x} = 2x - y$$

$$f_y = \frac{\partial f}{\partial y} = 2y - x$$

$$df = (2x - y)dx + (1y - x)dy$$

b. $f(x,y) = \frac{xy}{x^2 + y^2}$

$$d(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)y - (xy)(2x)}{(x^2+y^2)^2} = \frac{x^2y + y^3 - 2x^2y}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)x - (xy)(2y)}{(x^2+y^2)^2} = \frac{x^3 + xy^2 - 2x\ y^2}{(x^2+y^2)^2} = \frac{x^3 - x\ y^2}{(x^2+y^2)^2}$$

$$df = \left[\frac{y^3 - x^2y}{(x^2+y^2)^2}\right]dx + \left[\frac{x^3 - xy^2}{(x^2+y^2)^2}\right]dy$$

$$df = \frac{1}{(x^2+y^2)^2}\left[(y^3 - x^2y)dx + (x^3 - xy^2)dy\right]$$

c. $f(x,y) = e^{xy}\cos(x+y)$

$$df\ (x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial fdy}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = -e^{xy}\operatorname{sen}(x+y)(1) + \cos(x+y)ye^{xy}$$

$$\frac{\partial f}{\partial x} = e^{xy}(-\operatorname{sen}(x+y) + y\cos(x+y))$$

$$\frac{\partial f}{\partial y} = -e^{xy}\operatorname{sen}(x+y)(1) + \cos(x+y)xe^{xy}$$

$$\frac{\partial f}{\partial y} = e^{xy}(-\operatorname{sen}(x+y) + x\cos(x+y)$$

Luego:

$$df = e^{xy}(-\operatorname{sen}(x+y) + y\cos(x+y))dx + e^{xy}(x\cos(x+y) - \operatorname{sen}(x+y))dy$$

$$df = e^{xy}\left[(y\cos(x+y) - \operatorname{sen}(x+y)dx + (x\cos(x+y) - \operatorname{sen}(x+y)dy\right]$$

d. $\mu(x,y,z) = (xy + \frac{x}{y})^z$

$$d\mu = \frac{\partial \mu}{\partial x}dx + \frac{\partial \mu}{\partial y}dy + \frac{\partial \mu}{\partial z}dz$$

$$\frac{\partial \mu}{\partial x} = z(xy + \frac{x}{y})^{z-1}(y + \frac{1}{y}) = z(xy + \frac{x}{y})^{z-1}(\frac{y^2+1}{y})$$

$$\frac{\partial\mu}{\partial y}=z(xy+\frac{x}{y})^{z-1}(x-\frac{x}{y^2})=z(xy+\frac{x}{y})^{z-1}(\frac{xy^2-x}{y^2})$$

$$\frac{\partial\mu}{\partial y}=\frac{xz(y^2-1)}{y^2}(xy+\frac{x}{y})^{z-1}$$

$$\frac{\partial\mu}{\partial z}=(xy+\frac{x}{y})^{z}Ln(xy+\frac{x}{y})(1)=(xy+\frac{x}{y})^{z}Ln(xy+\frac{x}{y})$$

Luego:

$$d\mu=(xy+\frac{x}{y})^{z-1}(z\frac{(y^2+1)}{y}dx+\frac{xz(y^2-1)}{y^2}dy+(xy+\frac{x}{y})Ln(xy+\frac{x}{y})dz)$$

2. **Si** $z=e^{z}\cos x$. **Hallar** d^3z

SOLUCIÓN:

$$d^3z=(\frac{\partial}{\partial x}dy+\frac{\partial}{\partial y}dy)^3 z$$

$$d^3z=(\frac{\partial^3}{\partial x^3}dx^3+\frac{3\partial^2}{\partial x^2}\frac{\partial}{\partial y}dx^2dy+\frac{3\partial\partial^2}{\partial x\partial y^2}dxdy^2+\frac{\partial^3}{\partial y^3}dy^3)z$$

$$=(\frac{\partial^3}{\partial x^3}dx^3+\frac{3\partial^3}{\partial x^2\partial y}dx^2dy+\frac{3\partial^3}{\partial x\partial y^2}dxdy^2+\frac{\partial^3}{\partial y^3}dy^3)z$$

$$=(\frac{\partial^3 z}{\partial x^3}dx^3+\frac{3\partial^3 z}{\partial x^2\partial y}dx^2dy+\frac{3\partial^3 z}{\partial x\partial y^2}+\frac{\partial^3 z}{\partial y^3}dy^3)z$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}=-e^{y}senx \quad ; \quad \frac{\partial^2 z}{\partial x^2}=-e^{y}\cos x \quad ; \quad \frac{\partial^3 z}{\partial x^3}=e^{y}senx$$

$$\frac{\partial z}{\partial y}=e^{y}\cos x,\frac{\partial^2 z}{\partial y^2}=e^{y}\cos x,\frac{\partial^3 z}{\partial y^3}=e^{y}\cos x$$

$$\frac{\partial^3 z}{\partial x^2\partial y}=\frac{\partial^2}{\partial x^2}(\frac{\partial z}{\partial y})=\frac{\partial^2}{\partial x^2}(e^{y}\cos x)=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(e^{y}\cos x)\right]$$

$$=\frac{\partial}{\partial x}(-e^{y}senx)=-e^{y}\cos x$$

$$\frac{\partial^3 z}{\partial x\,\partial y^2}=\frac{\partial}{\partial x}(\frac{\partial^2 z}{\partial y^2})=\frac{\partial^2}{\partial x}(e^{y}\cos x)=-e^{y}senx$$

$$d^3z=(e^{y}senxdx^3-3e^{y}\cos xdx^2dy-3e^{y}senxdxdy^2+e^{y}\cos xdy^3)$$

$$d^3z=e^{y}(senxdx^3-3\cos xdx^2dy-3senxdxdy^2+\cos xdy^3)$$

3. Hallar el valor aproximado utilizando diferenciales de: $\sqrt{(5{,}02)^2+(1{,}99)^2+(5{,}97)^2}$

SOLUCIÓN:

$$f(x+\Delta x,y+\Delta y,z+\Delta z)=\sqrt{(3+0{,}02)^2+(2-0{,}001)^2+(6-0{,}03)^2}$$

Sea $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

Tomar:

$$x = 3 \quad ; \quad dx = 0{,}02$$
$$y = 2 \quad ; \quad dy = -0{,}01$$
$$z = 6 \quad ; \quad dz = -0{,}03$$

$$f(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \quad ; \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \quad ; \quad \frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

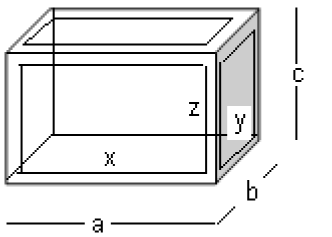
$$df(x,y,z) = \frac{x}{\sqrt{x^2+y^2+z^2}}dx + \frac{y}{\sqrt{x^2+y^2+z^2}}dy + \frac{z}{\sqrt{x^2+y^2+z^2}}dz$$

$$df(3,2,6) = \frac{3}{\sqrt{49}}(0{,}02) + \frac{2}{\sqrt{49}}(-0{,}01) + \frac{6}{\sqrt{49}}(-0{,}03) = -0{,}02$$

$$\Delta f \cong df \Rightarrow \Delta f(x,y,z) \cong df(x,y,z)$$
$$f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) \cong df(x,y,z)$$
$$f(x+\Delta x, y+\Delta y, z+\Delta z) = \sqrt{(3+0{,}02)^2 + (2-0{,}001)^2 + (6-0{,}03)^2}$$
$$= \sqrt{49} + (-0{,}02) = 6{,}98$$

4. **Se desea embalar un televisor cuyas dimensiones son 55cm de largo, 40cm de ancho y 80cm de altura con un material homogéneo cuyo peso es de $\frac{1}{20}\,gr/cm^3$ si el grosor del embalaje lateral es de 5cm mientras que de la base y la parte superior de 2,5cm cada uno. Usando diferenciales calcular aproximadamente es peso de la envoltura.**

SOLUCIÓN:



Sea: $x =$ el largo de la parte interior del volumen
$y =$ el ancho de la parte interior del volumen
$z =$ la altura de la parte interior del volumen

$$a = 55 + 2(5) = 65cm$$
$$b = 40 + 2(5) = 60cm$$
$$c = 80 + 2(3{,}5) = 85cm$$

$$V = xyz$$
$$dV \cong \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$
$$dV = yzdx + xzdy + xydz$$
$$dV = (40)(80)(10) + (55)(80)(10) + 55(40)(5)$$

$$\mathbf{Pe = dV \times Pc / n = [(40)(80)(10) + (55)(80)(10) + 55(40)(5)]\frac{1}{20}}$$

$$\mathbf{Pe = 1600 + 2200 + 550 = 4350}$$

DERIVACIÓN DE FUNCIONES COMPUESTAS

Para funciones diferenciales de una variable real la llamada regla de la cadena es un método para derivar una composición de funciones.

En efecto:

Si $y = f(x)$ **con** $u = g(x)$ **entonces:** $y = (f \circ g)(x) = f[g(x)]$

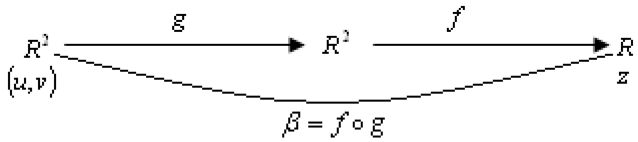
Luego: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'[g(x)]g'(x)$

Ahora veremos la regla de la cadena para funciones de dos variables donde cada una de estas variables es a su vez función de otras 2 variables.

Así consideramos las funciones:

$g : R^2 \to R^2 \quad \wedge \quad f : R^2 \to R$ **donde** $g(u,v) = (x,y) \quad f(x,y) = z$

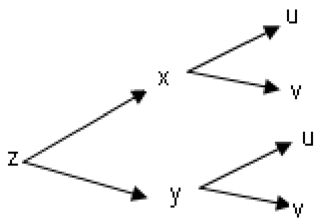
Al hacer la composición de estas funciones obtenemos una nueva función $\beta = f \circ g$ **definido por** $\beta = f[g(u,v)] = z$



Como z **es función** $\beta(u,v) = z$ **es una función de dos variables podemos hallar las derivadas parciales de** z **con respecto a las variables** u **y** v **mediante las siguiente regla de la cadena.**

Sea $f : S \subset R^2 \to R$ **una función definida por** $z = f(x,y)$ **tal que** $x = g(u,v); y = f(u,v)$ **son funciones diferenciales con respecto a las variables** u **y** v**.**

Si las derivadas parciales $\frac{\partial z}{\partial x}; \frac{\partial z}{\partial y}; \frac{\partial x}{\partial u}; \frac{\partial x}{\partial v}; \frac{\partial y}{\partial u}; \frac{\partial y}{\partial v}$ **existen, entonces,** z **es función de las variables** u **y** v **tal como se observa en el siguiente diagrama.**

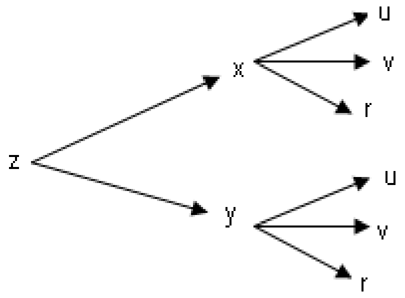


Entonces las derivadas parciales de z **con respecto a las variables** u **y** v **está dado por:**

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad ; \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

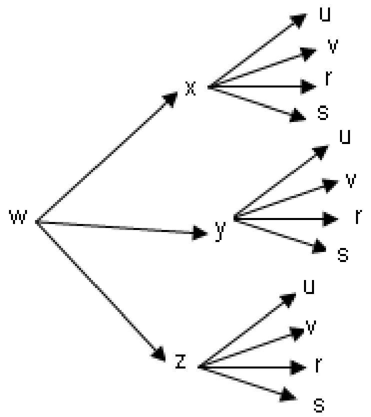
OBSERVACIÓN:

1. **En el siguiente diagrama el exponente del dominio de la función g indica el número de ecuaciones diferenciales y el exponente del rango de la función g indica el número de términos que debe tener cada ecuación diferencial en el segundo miembro así por ejemplo:**



$$\underset{(u,v,r)}{R^3} \longrightarrow \underset{(x,y)}{R^2} \longrightarrow \underset{z}{R}$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad ; \quad \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \quad ; \quad \frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$



$$\underset{(u,v,r,s)}{R^4} \longrightarrow \underset{(x,y,z)}{R^3} \longrightarrow \underset{w}{R}$$

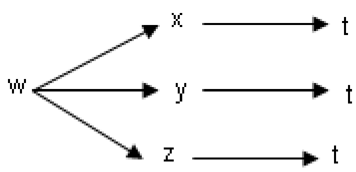
$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad ; \quad \frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \quad ; \quad \frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

DEFINICIÓN:

Sea $f : S \subset R^3 \to R$ una función definida por $w = f(x, y, z)$ tal que $x = g_1(t);\quad y = g_2(t);\quad z = g_3(t)$ son funciones diferenciales respecto a la variable t en este caso en lugar de una derivada parcial se obtiene una derivada total.

En efecto:



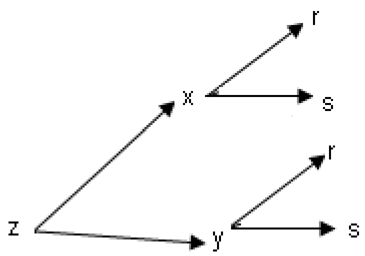
$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

EJEMPLOS:

1. **Por la regla de la cadena hallar las derivadas correspondientes de las siguientes funciones:**
   - **a.** $z = e^{y/x}$; $x = 2r\cos s$; $y = 4r\,\text{sens}$
   - **b.** $z = x^2 y$; $x = \cos t$; $y = \text{sent}$
   - **c.** $w = \sqrt{x^2 + y^2 + z^2}$; $x = e^r \cos s$; $y = e^r\,\text{sens}$; $z = e^s$
   - **d.** $z = Ln(x^2 + y^2) + \sqrt{x^2 + y^2}$; $x = e^t \cos t$; $y = e^t\,\text{sent}$
   - **e.** $\mu = x^3 y$; $x^5 + y = t$; $x^2 + y^3 = t^2$

SOLUCIÓN

a. $z = e^{y/x}$; $x = 2r\cos s$; $y = 4r\,\text{sens}$



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = -\frac{y}{x^2} e^{y/x}(2\cos s) + \frac{1}{x} e^{y/x}(4\,\text{sens})$$

$$\frac{\partial z}{\partial r} = \frac{1}{x} e^{y/x}\left[4\,\text{sens} - \frac{y}{x}(2\cos s)\right]$$

$$\frac{\partial z}{\partial r} = \frac{e^{\frac{4r\,\text{sens}}{2r\cos s}}}{2r\cos s}\left[4\,\text{sens} - \frac{4r\,\text{sens}}{2r\cos s}(2\cos s)\right]$$

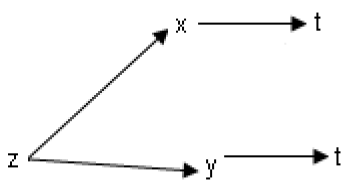
$$\frac{\partial z}{\partial r} = \frac{e^{2\tan s}}{2r\cos s}\left[4\,\text{sens} - 4\,\text{sens}\right] = 0$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = -\frac{y}{x^2} e^{y/x}(-2r\,\text{senss}) + \frac{1}{x} e^{y/x}(4r\,\text{sens})$$

$$\frac{\partial z}{\partial s} = \frac{1}{x} e^{y/x}\left[4r\cos s - \frac{y}{x}(2r\,\text{sens})\right] = \frac{e^{2\tan s}}{2r\cos s}\left[4r\cos s + \frac{4r\,\text{sens}}{2r\cos s}(2r\,\text{sens})\right]$$

$$\frac{\partial z}{\partial s} = \frac{e^{2\tan s}}{2\cos s}\left[2\cos s + \frac{2\,\text{sen}^2 s}{2\cos s}\right] = 2\,\frac{e^{2\tan s}}{\cos^2 s}$$

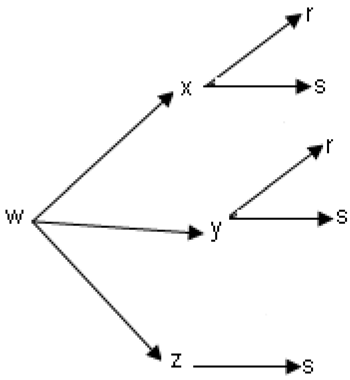
b. $z = x^2 y$; $x = \cos t$; $y = \text{sent}$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = -2xysent + x^2 \cos t = -2(\cos t)(sen^2 t) + \cos^2 t(\cos t) = \cos t(\cos^2 t - 2sen^2 t)$$

c. $w = \sqrt{x^2 + y^2 + z^2}$; $x = e^r \cos s$; $y = e^r sens$; $z = e^s$



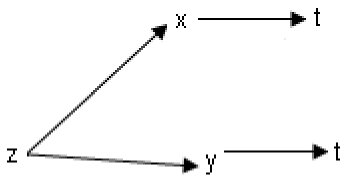
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}\left[x\frac{\partial x}{\partial r} + y\frac{\partial y}{\partial r}\right]$$

$$\frac{\partial w}{\partial r} = \frac{1}{\sqrt{e^{2r} + e^{2s}}}(e^r \cos s.e^r \cos s + e^r sens - e^r sens) = \frac{e^{2r}}{\sqrt{e^{2r} + e^{2s}}}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}\left[x\frac{\partial x}{\partial s} + y\frac{\partial y}{\partial s} + z\frac{\partial z}{\partial s}\right]$$

$$\frac{\partial w}{\partial s} = \frac{1}{\sqrt{e^{2r} + e^{2s}}}(e^r \cos s(-e^r sens) + e^r sens(e^r \cos s) + e^s e^s) = \frac{e^{2s}}{\sqrt{e^{2r} + e^{2s}}}$$

d. $z = Ln(x^2 + y^2) + \sqrt{x^2 + y^2}$; $x = e^t \cos t$; $y = e^t sent$



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2} + \frac{x}{\sqrt{x^2 + y^2}} = \frac{2e^t \cos t}{e^{2t}} + \frac{e^t \cos t}{e^t} = \cos t.(\frac{2}{e^t} + 1)$$

$$\frac{dx}{dt} = -e^t sent + e^t \cos t = e^t(\cos t - sent)$$

$$\frac{\partial z}{\partial y} = \frac{2x}{x^2 + y^2} + \frac{y}{\sqrt{x^2 + y^2}} = \frac{2e^t sent}{e^{2t}} + \frac{e^t sent}{e^t} = sent\,(\frac{2}{e^t} + 1)$$

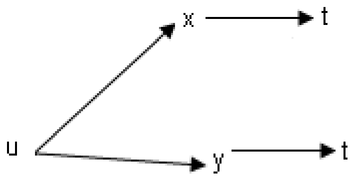
$$\frac{dy}{dt} = e^t \cos t + e^t sent = e^t(\cos t + sent)$$

$$\frac{dz}{dt} = \cos t(\frac{2}{e^t} + 1)e^t(\cos t - sent) + sent(\frac{2}{e^t} + 1)e^t(\cos t + sent)$$

$$\frac{dz}{dt} = e^t(\frac{2}{e^t} + 1)[\cos t(\cos t - sent) + sent(\cos t + sent)]$$

$$\frac{dz}{dt} = e^t\left(\frac{2+e^t}{e^t}\right)(\cos^2 t + \operatorname{sen}^2 t) = 2 + e^t$$

e. $\mu = x^3 y$; $x^5 + y = t$; $x^2 + y^3 = t^2$



$$\frac{d\mu}{dt} = \frac{\partial\mu}{\partial x}\frac{dx}{dt} + \frac{\partial\mu}{\partial y}\frac{dy}{dt} = 3x^2y\frac{dx}{dt} + x^3\frac{dy}{dt} \text{ ....(*)}$$

$$\begin{matrix}(2) \\ (-5x^3)\end{matrix}\begin{cases} 5x^2\dfrac{dx}{dt} + \dfrac{dy}{dt} = 1 \\ 2x\dfrac{dx}{dt} + 3y^2\dfrac{dy}{dt} = 2 \end{cases} \Rightarrow \quad \begin{aligned} 10x^4\frac{dx}{dt} + 2\frac{dy}{dt} &= 2 \\ -10x^4\frac{dx}{dt} - 15x^3y^2\frac{dy}{dt} &= -10x^3t \end{aligned}$$

$$\left(2 - 15x^3y^2\right)\frac{dy}{dx} = 2 - 10x^3t$$

$$\frac{dy}{dt} = \frac{2 - 10x^3t}{2 - 15x^3y^2} \Rightarrow \frac{dy}{\partial t} = \frac{10x^3t - 2}{15x^3y^2 - 2}$$

$$\begin{cases} -15x^4y^2\dfrac{dx}{dt} - 3y^2\dfrac{dy}{dt} = -3y^2 \\ 2x\dfrac{dx}{dt} + 3y^2\dfrac{dy}{dt} = 2t \end{cases}$$

$$(2x - 15x^4y^2)\frac{dx}{dt} = 2t - 3y^2$$

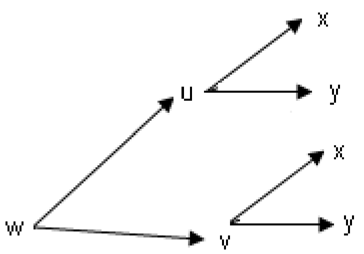
$$\frac{dx}{dt} = \frac{3y^2 - 2t}{15x^4y^2 - 2x}$$

En (*)

$$\frac{d\mu}{dt} = 3x^2y\left(\frac{3y^2 - 2t}{15x^4y^2 - 2x}\right) + x^3\left(\frac{10x^3t - 2}{15x^3y^2 - 2}\right) = \frac{3xy(3y^2 - 2t) + x^3(10x^2t - 2)}{15x^3y^2 - 2}$$

2. **Si $w = f(x^2 - y^2; y^2 - x^2)$. Hallar $\left(y\dfrac{\partial w}{\partial x} + x\dfrac{\partial w}{\partial y}\right)$**

SOLUCIÓN



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = f_u(u,v)2x + f_v(u,v)(-2x) = 2xf_u - 2xf_v$$

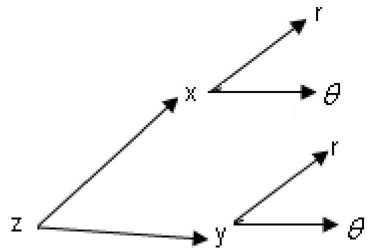
$$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}= f_u(u,v)(-2y)+ f_v(u,v)(2y)=-2yf_u+2yf$$

$$\left(y\frac{\partial w}{\partial x}+x\frac{\partial w}{\partial y}\right)=y(2xf_u-2xf_v)+x(-2yf_u+2yf)=0$$

3. **Si** $z = f(x,y)$ **es diferenciable en** x, y **; si** $x = r\cos\theta$ **;** $y = r\operatorname{sen}\theta$ **Hallar:**

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

SOLUCIÓN



$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\operatorname{sen}\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2=\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\operatorname{sen}\theta\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+2\cos\theta\operatorname{sen}\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+\left(\frac{\partial z}{\partial y}\right)^2\operatorname{sen}^2\theta$$

$$\frac{\partial z}{\partial\theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial\theta}=\frac{\partial z}{\partial x}(-r\operatorname{sen}\theta)+\frac{\partial z}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial z}{\partial\theta}\right)^2=\left[\frac{\partial z}{\partial x}(-r\operatorname{sen}\theta)+\frac{\partial z}{\partial y}(r\cos\theta)\right]^2$$

$$\left(\frac{\partial z}{\partial\theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2r^2\operatorname{sen}^2\theta-2r^2\cos\theta\operatorname{sen}\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+\left(\frac{\partial z}{\partial y}\right)^2r^2\cos^2\theta$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\operatorname{sen}^2\theta-2\cos\theta\operatorname{sen}\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+\left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta$$

Por lo tanto:

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+2\cos\theta\operatorname{sen}\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+\left(\frac{\partial z}{\partial y}\right)^2\operatorname{sen}^2\theta+$$
$$\left(\frac{\partial z}{\partial x}\right)^2\operatorname{sen}^2\theta-2\cos\theta\operatorname{sen}\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+\left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

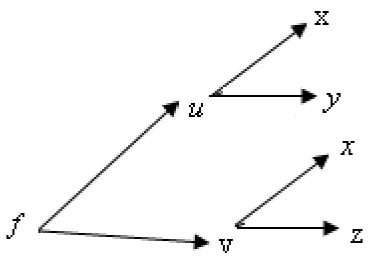
4. **Si** $w=\frac{xy}{z}\ln x+xf\left(\frac{y}{x};\frac{z}{x}\right)$ **hallar:** $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}$

SOLUCIÓN

$$\frac{\partial w}{\partial x}=\frac{xy}{z}\frac{1}{x}+\frac{y}{z}\ln x+x\frac{\partial}{\partial x}\,f\left(\frac{y}{x},\frac{z}{x}\right)+\,f\left(\frac{y}{x},\frac{z}{x}\right)\frac{\partial x}{\partial x}$$

$$\frac{\partial w}{\partial x}=\frac{y}{z}+\frac{y}{z}\ln x+\,f\left(\frac{y}{x},\frac{z}{x}\right)+x\frac{\partial}{\partial x}\,f\left(\frac{y}{x},\frac{z}{x}\right)\text{..................}(*)$$

Sea $u=\frac{y}{x}\qquad\wedge\qquad v=\frac{z}{x}$



$$\frac{\partial f\left(\frac{y}{x},\frac{z}{x}\right)}{\partial x}=\frac{\partial f}{\partial x}\left(-\frac{y}{x^2}\right)+\frac{\partial f}{\partial v}\left(-\frac{z}{x^2}\right)=-\frac{y}{x}\,f_u-\frac{z}{x}\,f_v$$

Por lo tanto:

$$\frac{\partial w}{\partial x}=\frac{y}{z}+\frac{y}{z}\ln x+\,f\left(\frac{y}{x},\frac{z}{x}\right)+x\left(-\frac{y}{x}\,f_u-\frac{z}{x}\,f_v\right)$$

$$\frac{\partial w}{\partial x}=\frac{y}{z}+\frac{y}{z}\ln x+\,f\left(\frac{y}{x},\frac{z}{x}\right)-yf_u-zf_v$$

<u>APLICACIÓN DE LAS DERIVADAS PARCIALES</u>

EJEMPLOS:

1. **Hallar la ecuación del plano tangente a la esfera $x^2 + y^2 + z^2 = 1$ que es paralelo al plano $P = 8x + 6y + 10z = 1$**

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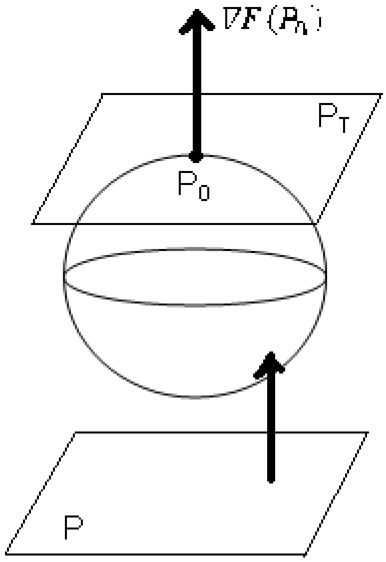

$$P_T : (P - P_0).\nabla F(P_0) = 0$$

$$P_T \mathbin{//} P \Rightarrow \nabla F(P_0) \mathbin{//} \vec{N} \Rightarrow \nabla F(P_0) = K\vec{N} \text{..........................}(*)$$

$$\vec{N} = (8,6,10)$$

$$F(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$$

$$\nabla F(x,y,z) = (2x,2y,2z) = 2(x,y,z)$$

$$\nabla F(P_0) = 2(x_0, y_0, z_0)$$

En (*):

$$2(x_0, y_0, z_0) = 2k(4,3,5)$$

$$(x_0, y_0, z_0) = (4k,3k,5k)$$

$$x_0 = 4k$$

$$y_0 = 3k$$

$$z_0 = 5k$$

Como $P_0 \in F(x,y,z)$ **se cumple que:**

$$F(x_0, y_0, z_0) = x_0^2 + y_0^2 + z_0^2 - 1 = 0$$

$$(4k)^2 + (3k)^2 + (5k)^2 = 1 \quad \Rightarrow \quad 50k^2 = 1 \quad \Rightarrow \quad k = \pm\frac{1}{5\sqrt{2}}$$

Entonces: $x_0 = \pm\frac{4}{5\sqrt{2}}$; $y_0 = \pm\frac{3}{5\sqrt{2}}$; $z_0 = \pm\frac{5}{5\sqrt{2}}$

$$\nabla F(P_0) = 2(x_0, y_0, z_0) = 2\left(\pm\frac{4}{5\sqrt{2}}; \pm\frac{3}{5\sqrt{2}}; \pm\frac{5}{5\sqrt{2}}\right) = \pm\frac{1}{5\sqrt{2}}(8,6,10)$$

$$P_T : \left[(x, y, z) - \left(\pm\frac{4}{5\sqrt{2}} ;\ \pm\frac{3}{5\sqrt{2}} ;\ \pm\frac{5}{5\sqrt{2}} \right) \right] \pm \frac{1}{5\sqrt{2}}(8,6,10) = 0$$

$$\left(x \pm \frac{4}{5\sqrt{2}} ; y \pm \frac{3}{5\sqrt{2}} ; z \pm \frac{5}{5\sqrt{2}} \right)(8,6,10) = 0$$

$$8x \pm \frac{32}{5\sqrt{2}} + 6y \pm \frac{18}{5\sqrt{2}} + 10z \pm \frac{50}{5\sqrt{2}} = 0$$

$$8x + 6y + 10z = \pm\frac{100}{5\sqrt{2}} \quad \Rightarrow \quad 4x + 3y + 5z = \pm\frac{10}{\sqrt{2}}$$

2. **Hallar la ecuación normal al elipse $x^2 + 2y^2 + 3z^2 = 1$ que es paralelo a la recta**
   $R:\ x = 3t + 1;\quad y = 2t;\quad z = 3t - 1$

SOLUCIÓN